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ACTUARIAL SOCIETY OF AMERICA

"The work of science is to substitute facts for appearances and demonstrations for impressions."—RUSKIN.

TRANSACTIONS

1913

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NOTICE.

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OCTOBER, 1913.

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NOTATION

Life Symbols.

As Revised October 11th, 1911.

Resolved, that in the presentation of papers the symbols of the Text-Book of the Institute of Actuaries, with the additions noted below, be followed when convenient, and that, in any event, the use of such symbols in other senses be avoided; all letters used as symbols to be printed in italics except "a" when used to denote an annuity of 1 first payment immediate, *i. e.* $(1+a)$; that N_x and S_x be used in accordance with the practice of the Institute of Actuaries, *i. e.*,

$$N_x = N_{x-1} = D_x + D_{x+1} + \cdots$$
$$S_x = S_{x-1} = N_x + N_{x+1} + \cdots$$

Additional symbols

$$k_x = \frac{d_x}{l_{x+1}} = \frac{C_x}{D_{x+1}}.$$

$$u_x = \frac{l_x}{l_{x+1}}(1+i) = \frac{D_x}{D_{x+1}}.$$

Disability Symbols.

As Adopted May 16th, 1912.

l_x^{aa} = the number of active lives at age x.
l_x^{ii} = the number of invalid lives at age x,

then $l_x^{aa} + l_x^{ii} = l_x$.

l_x^{i} = the survivors at age x in a mortality table based on invalid lives (not to be confused with l_x^{ii} amongst whom there are additions each year from those just disabled).

d_x^{aa} = the number dying as active lives between ages x and $x+1$.

d_x^{ii} = the number dying as invalid lives between ages x and $x+1$.

i_x = the number of active lives becoming invalid between ages x and $x+1$.

p_x^a = the probability of an active life aged x being alive one year hence, whether then active or invalid.

q_x^a = the probability of an active life aged x dying within a year, whether still active or after becoming invalid,

$$\text{then } p_x^a + q_x^a = 1.$$

p_x^i = the probability of an invalid life aged x being alive one year hence.

q_x^i = the probability of an invalid life aged x dying within a year,

$$\text{then } p_x^i + p_x^i = 1.$$

p_x^{aa} = the probability of an active life aged x being alive and active one year hence.

p_x^{ai} = the probability of an active life aged x being alive but invalid one year hence.

q_x^{aa} = the probability of an active life aged x dying while still active within a year.

q_x^{ai} = the probability of an active life aged x becoming invalid and dying within a year.

r_x = the probability of an active life aged x becoming invalid within a year,

then $p_x^{aa} + p_x^{ai} = p_x^a$; $q_x^{aa} + q_x^{ai} = q_x^a$; $p_x^{ai} + q_x^{ai} = r_x$; $p_x^{aa} + q_x^{aa} = 1 - r_x$; $p_x^{aa} + p_x^{ai} + q_x^{aa} + q_x^{ai} = 1$.

r_x' = the absolute annual rate of invalidity,

$$\text{then } r_x' = \frac{i_x}{l_x^{aa} - \frac{1}{2}d_x^{aa}} = \frac{r_x}{1 - \frac{1}{2}q_x^{aa}}.$$

$D_x^{aa} = v^x l_x^{aa}$.

$D_x^{ii} = v^x l_x^{ii}$.

$D_x^i = v^x l_x^i$.

a_x^{ai} = value of an annuity payable at the end of each year

provided an active life now aged x is then alive but invalid.

$\mathbf{a}_x^a$ = value of an annuity-due on an active life payable during survival, active or invalid.

$\mathbf{a}_x^{aa}$ = value of an annuity-due on an active life payable during activity.

$\mathbf{a}_x^i$ = value of an annuity-due on an invalid life.

${}_n|\mathbf{a}_{x\,\overline{t}|}^{a\,(\overline{i\,n}|)}$ = value of a deferred temporary annuity payable at the beginning of each of t years deferred n years, provided the disability occur during the n years.

$\mathbf{a}_{x\,:\,\overline{n}|\,:\,\overline{y-x}|}^{aa}$ = value of an annuity-due on an active life for n years or for $y-x$ years, whichever is the shorter, during activity.

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TRANSACTIONS

MAY 15TH AND 16TH, 1913.

Address of the President, William C. Macdonald. A Review.

"What of the near future, what of the far future?" was the question suggested by your esteemed President, Mr. Weeks, in his address to the Society in 1906, in reference to the insurance legislation which had been then recently enacted in the State of New York. While we may be unable to forecast the future it may not be without interest to consider, in view of the length of time which has elapsed, some of the effects of this legislation and present day tendencies of the business. In presuming to offer these few comments I must plead in extenuation the important influence exercised by this State on the business in which we are engaged and also by some of the large companies with head offices here whose sphere of operations is not limited by international boundaries.

We all remember the public interest which the appointment of the Armstrong Commission created, and which continued throughout, in its proceedings and until the legislature had ceased its labors and the new Act had become law. The magnitude of the financial interests involved and the irregularities revealed would not alone have been sufficient to create the great interest manifested. The explanation is rather to be found in the fact that the influence of the business and the interests which it serves extend from the highest through all grades of business and social life to that of the cottager and artizan. Few businesses other than those which furnish the actual necessities of life come more in touch

with the needs of the great mass of the people or bear in a greater degree upon their economic welfare.

This legislation has been reviewed by many, criticized by some, and commended by others, but whatever may be our individual views, few will be found to deny that the members of the Commission were actuated by a sincerity of purpose and a high desire to do that which, in their judgment, would most fully protect the policyholder and promote the best interests of the business. Legislation, however, which is enacted hastily and under great public excitement, is liable to be faulty or extreme, rendering subsequent amendment or modification necessary. The present was no exception to the rule. Numerous changes have already been made therein, its important principles have, however, been adhered to with results which have been, I think, in many ways beneficial.

Immediately following the New York State Investigation, in fact almost simultaneously therewith, investigations were instituted in other states and in Canada, which resulted in a large amount of new insurance legislation being placed upon the statute books, some of it wise, much of it otherwise, often sectional in character, restrictive in its nature, interfering unduly with the operations of the companies and the freedom of contract, and consequently disadvantageous to the best interests of the policyholder. This condition, which continued more especially throughout the years 1906, 1907 and 1908, has extended in a lesser degree to the present time. It arises in a large measure out of the general antagonism which obtains to-day towards all corporations on the part of the general public. Legislation imposing heavy administrative burdens upon the companies unnecessarily and without corresponding advantage is all too common.

The heaviest and most unfair burden which is imposed upon life insurance companies to-day is in the form of governmental taxation, the tendency of which is to become greater from year to year. In opposing or in seeking redress from these measures our arguments and appeals have been made to the legislator, their framer and creator. We have overlooked almost entirely the policyholder. Is it not to the latter that we should in a careful and systematic manner present our arguments, and seek by reasonable and fair means to impress upon him the fact that the burden of all unnecessary legislation and taxation must ultimately be borne by him, either directly in the form of increased premium rates, or indi-

rectly in reduced dividends? To be effective this must be done systematically. Much useful educational work could be accomplished by every company without much added expense, either through our annual reports, a page or more of which might be devoted to this purpose, or through the medium of the journals published by many companies, or by means of other literature issued from time to time.

An outstanding result of the investigation has been the impetus given to the organization of new companies, both in the United States and Canada. At the beginning of 1906 the active life companies reported as operating in the United States numbered one hundred and thirty-two. In 1912 they had increased to two hundred and fifty-two, or very nearly doubled. The increase in Canada has not been relatively so great. The contributing causes are various. The limitation placed on the volume of business a company was permitted to write under the New York law opened up to some extent a field for new enterprise. Field men whose positions had been endangered during the period of reorganization following the investigation, sought new spheres for their energies and abilities, while in other cases local ambitions or necessities were the actuating causes. Much prominence had been given to the value of the stock in some old and well-established companies and the rate of dividend paid. This was utilized by promoters, often unfairly, in securing subscriptions for new shares. I can speak with authority in regard to Canada. The conditions under which the business of these old companies was established were vastly different from those of today. The amount of the paid-up capital was usually relatively small, and the rate of dividend now paid has been given much prominence without regard to the experience or record of the past. The stock of these new companies is usually issued at a premium varying from 25% to 100%, a large portion of which is necessarily absorbed in flotation expenses. Some of them give evidence of capable and energetic management, and we wish all of them every success, but it is safe to say that a number of them will be absorbed by other companies. Instances of this have already occurred. The interests of the policyholder under the present stringent laws will doubtless be safeguarded, but the stockholders will, in many cases, experience some disappointment. While a bar cannot be placed on the organization of new companies, however beneficial it might be to have a "close" or "holiday" season for a period in life insur-

ance, yet, if insurance legislation is desirable, some provision regulating the issue of prospectuses and of estimated stock dividend results might not prove unwholesome.

Possibly no result of the investigation was more to be regretted than the sacrifice of a large volume of insurance protection, due to the unrest and want of confidence created in the public mind, which was aggravated unduly by sensational and often unfair articles in certain sections of the daily press, an unnecessary sacrifice, for the irregularities revealed were functional only, and not organic. They were susceptible of treatment and easily cured. The business was sound, and it is gratifying to record that the financial status of all the old, established companies in the United States and Canada and their ability to fulfil their obligations to the uttermost could never be honestly or seriously questioned. The decrease in the volume of new business written, which amounted to as much as sixty per centum, was partly due to the limitation imposed upon the amount which a company might write, the temporary disorganization of the agency staffs, and the financial panic of 1907, the effects of which extended into 1908. The business terminated by lapse and surrender in the companies reporting to the New York Insurance Department in 1905 was equal to 6.77 per centum, and in 1906 it was 7.29 per centum of the total amount at risk, the business terminated by lapse alone in the latter year being 20.2 per centum of the total business issued in the two preceding years. Public confidence, however, was soon restored, and a marked improvement was evidenced in the succeeding years in the volume of business written and in the lapse rate, which has continued to the present time. The lapsed business in the past three years has not averaged more than twelve per centum of the amount of new insurances issued in the two preceding years, as compared with a ratio of 13.7 per centum for the year 1904, and an average for the three years 1902, 1903 and 1904 of about 13 per centum. While the difference is not great it bears some evidence of improvement in the quality of the business now being placed on the books as compared with that of the preinvestigation years.

Undoubtedly the most important and far-reaching provision was that in which attempt was made to regulate the cost of new business and in prescribing in definite terms the maximum remuneration which might be paid to agents. In so far as I know this is the

first instance, on this continent at least, of sumptuary legislation in life insurance. The fact that the cost of the business was excessive and was becoming more so had been recognized for some time. The mad race for volume of business, which had obtained for some years, had assisted in bringing this about, but attempts made by voluntary arrangement on the part of the officers of the companies to call a halt always ended in failure. The personal equation entered too largely into the question. The ambition for volume was not confined exclusively to any particular class of companies. While it was an offence usually charged against the large companies, the smaller ones were also offenders. Combined with this was the pernicious practice of rebating, which had become so general, and which added not only to the cost of the business, but exercised a demoralizing influence alike on the policyholder and the agent. The limitation of the commissions allowable, reinforced by the laws passed in other states rendering rebating a punishable offence, have resulted in the elimination very largely of this abuse. I can speak advisedly in so far as Canada is concerned where it is to-day practically little heard of. The limitation of the commissions has operated most effectively in reducing the cost of the business, and this without injury to the welfare of the agent. Allowing one-quarter of one per cent. of the mean invested assets for investment expenses, and assessing the balance of the expenses against the first year and renewal premium income on the basis, respectively, of ten to one, the average cost of the new business of the New York State companies for the years 1910 and 1911 will be found to be 72 per centum of the premiums as compared with an average cost ratio for the years 1903 and 1904, immediately preceding the investigation, of 84.8 per centum, thus recording an improvement of 12 per centum. These ratios are determined upon a revenue basis. Tests applied in other ways will show relatively favorable results. This record has been accomplished during a period when the cost of living was increasing, when the purchasing power of the dollar was decreasing, and when we might consequently have expected an increase rather than a decrease in cost.

In Canada, where the investigation followed very closely the lines of that in New York and the legislation proposed was very similar, but owing to the facts that public feeling did not run so high and that some considerable time elapsed before the recommendations of the Royal Commission were considered by Parliament, it was found

possible to have many of the provisions of the proposed law modified or eliminated entirely, the proposal in regard to the limitation of expenses coming under the latter heading. Viewed in the light of present experience the wisdom of that action would appear doubtful, inasmuch as a careful analysis shows that the expenses of the business in Canada, measured on the foregoing basis, are on an ascending rather than a descending scale. A reasonable limit imposed upon the cost of obtaining new business, leaving the companies free as to the mode or basis of agency remuneration, might have proved beneficial.

The liberalizing of the policy contracts which has taken place in recent years has been most marked, and the present day contract with its entire absence of conditions and restrictions, its various non-forfeiture privileges and instalment options and disability benefits, etc., presents an interesting contrast to the contracts of even a quarter of a century ago. These added benefits and privileges are greatly to the advantage of the policyholder, but if government standardized policies had been adopted some years ago it will hardly admit of argument that the policyholder of to-day would not be in the enjoyment of as liberal a form of contract nor one embodying so many benefits as that which he is now entitled to receive. Improvement and progress in this respect would have been slower. The lesson is obvious. We should seek by legislation to safeguard merely without unduly restricting or interfering with the liberty of contract.

In one respect I am inclined to share the opinion entertained by some others and think that it is questionable if we have not gone too far, viz., in the granting of high cash surrender and loan values, the full reserve value being in some instances guaranteed after a policy has been but a few years in force. Doubtless this may be of advantage from a canvassing standpoint and facilitate the work of the agent by adding to the apparent attractiveness of the policy contract, but it unquestionably facilitates the "twisting" of business, and in the event of unfavorable circumstances, external or internal, which may have the effect of creating a want of confidence, must tend to increase withdrawals. We are in this way placing a temptation before a policyholder to surrender his contract and thus deprive those dependent upon him of the protection which he had thus provided for them. The policyholder who can withdraw the full reserve value of his policy will, under certain circumstances,

be undoubtedly and unnecessarily tempted to do so. We can all perhaps cite cases which have come under our observation in practice where a person holding several policies in the same or different companies has, after careful enquiry, decided to surrender the policy yielding the most favorable surrender value.

The high cash value carries with it the accompanying privilege of high loan values obtainable on demand and upon most reasonable terms, probably more favorable than can be otherwise secured. The effect of this is making itself felt in a marked degree at the present time. At the end of 1905 the total amount of the policy loans in companies reporting to the New York Department amounted to $196,710,000, equal to 8.8 per centum of the reserve values. In 1912 the loans had increased to $525,589,000, or equal to 15.2 per centum. In Canada at the end of 1905 the policy loans amounted to $9,679,000, or 10.6 per centum of the amount of the policy reserve. At the end of 1911, the figures for 1912 not being available, the policy loans had increased to $22,960,000, or 14.2 per centum.

The value to the policyholder of being able to borrow on demand and upon easy terms on his policy may not be questioned. Experience indicates, however, that a comparatively small amount of the loans secured are discharged before the maturity of the contract, and they not infrequently pave the way to the surrender of the policy. Is our liberality in this respect altogether wise?

The abolition of the deferred, and the compulsory adoption of the annual, dividend system was one of the important provisions of the New York law. The deferred dividend system no doubt had led to abuses, but they were not of a nature which it was impossible to remedy. Its retention, accompanied by a proper system of accounting and the prohibition of estimates, would have preserved a plan of insurance which appealed to many, and which possessed some distinct advantages, particularly in connection with the insurance of under-average or sub-standard lives.

The adoption exclusively of the annual dividend system by the State of New York and the general demand which has been created for this form of contract has, combined with the liberalization of the policy contract, resulted in a form of competition which may not prove altogether healthy, though it is admittedly to the credit of this system that it encourages economical management. Competition to-day is drifting almost entirely into a question of lowest net cost and highest cash surrender values. The desire, we might even

say the necessity, in order to meet competition, to declare large dividends, is already apparent. Danger is not likely to arise when financial conditions are favorable, as at present, when trade is good and the earning power of the companies is gradually improving. If a day of reckoning comes, and I would not wish to be considered pessimistic, it will be when financial and commercial depression ensue, when the earning power of money is reduced with consequent variations in the values of securities, and when losses on investments may occur or other unforeseen contingencies arise. The maintenance of substantial surplus reserves—greater than would be requisite under the deferred dividend system—is an important necessity in order to properly safeguard and protect the company in such an emergency, as the temptation to maintain the rate of dividend beyond that which prudence would justify will be great. Statutory limitations as to the amount of the surplus reserves a company may hold are not only unnecessary, but may be fraught with danger. The wise legislator will seek to encourage and not restrict the building up of surplus reserves within all reasonable limits. The keenness of present-day competition will impel us in the opposite direction and will be sufficient to ensure that the accumulations are not excessive. It may not be by any means the company which is paying the highest scale of dividends which will experience the greatest measure of strain. On the contrary, it is not unlikely that it will be greater in those somewhat lower down in order who are straining their resources that they may equal or approximate their competitors.

While the business to-day may present problems for careful and conservative consideration, yet there is good ground for high hope and confidence as to the future. It is better and freer from pernicious influences than it has been in the past. The benefits to the policyholder have been greatly increased, and insurance protection has been placed within the reach of many to whom it has hitherto been denied. Heretofore its chief care has been the protection of the widow and orphan, but to-day we see its strong right arm reaching out to assist by the application of educational and practical methods in the conservation and preservation of human life.

Charles Gill: The First Actuary in America.

BY

EMORY MCCLINTOCK.

At the last (October, 1912) meeting of the Actuarial Society of America, at which I was unable to be present, I took occasion to express in writing my surprise that so very little seemed to be known of the life and personality of Charles Gill, the first Actuary of the Mutual Life Insurance Company of New York, which is accustomed to reckon itself the earliest in the country in date of establishment. In saying this I would not throw doubt upon its claim to priority, which I have myself always upheld. I have taken the first opportunity to seek out and present such facts as I could find on the subject of Charles Gill.

Life in England.

Charles Gill was born at Staxton, in Yorkshire, England, according to a certificate which I recently procured from the Rev. Robert A. White, the vicar of the parish church at Willerby in England. The certificate is as follows:

"Baptisms from Lady Day, 1805, to Lady Day, 1806.

"No. 73rd. Oct. 22nd., A. D. 1805, Charles, son of David and Ann Gill of Staxton, shoemaker."

The first life policy in the Mutual Life on Mr. Gill was paid on November 26, 1855, a date which came as nearly as possible fifty years from his birth. It was issued for $2,000 on the 12th of March, 1850, in favor of his wife, Eliza E. Gill. Another policy was issued exactly two years later, March 12, 1852. This was paid on the same day as the first to his three children. Their names were Eliza Anne Gill, William C. Gill, and Mary Gill. The amount of this second policy was $3.000. The date of Mr. Gill's birth was stated by himself, in his first application, as 22 October, 1805. As his evidence is not exact, the date being the same as the day of his baptism, as may be seen from the preceding certificate, it is to be presumed that about three days intervened between his birth and his baptism, in which case he must have been born three days, or

more, before 22 Oct., 1805. The account books of the Mutual Life give the date of his death as being his fiftieth birthday. Walford's Cyclopædia (Insurance) states his death to have been on the same date, but apparently through a misprint states he was "aged 51." The same date is also given in a memoir of him, appearing in the *Journal of the Institute of Actuaries,* 1856, pp. 216–227. But the mortuary record of the Mutual Life, which is the best official record of his death, makes it as occurring on 26 Oct., 1855. It seems established in any case, that he lived half a century.

There are several sources from which a few facts can be gathered about his early life, and his subsequent career in America. The president of the Mutual Life, Frederick S. Winston, contributed some of these, and a warm friend of Gill's wrote at much greater length, namely, the Rev. W. A. Muhlenberg, D.D., the noted president of St. Paul's College at Flushing, where Gill had the mathematical chair during several years. Both writers aided the editor of the *United States Insurance Gazette* in publishing such facts as lay in their power, with the result of providing a biographical sketch of considerable value, which was at one time copied by the *Assurance Magazine* (*J. I. A.*) of London, which has already been referred to. Both writers will be drawn upon here.

Winston says of his early life: "he left his village school at ten years of age, his schoolmaster declaring that he knew as much as he did. After this all his knowledge was procured by his own unaided exertions."

Dr. Muhlenberg's statement is more complete: "It appears from the testimony of his daughter, that her father was born Oct. 22, 1805, in a small, secluded country village, in the midland county of Yorkshire, England. Although the name of the village is not given, still we can form some idea of its obscurity and lack of educational institutions from the fact that it had neither 'school, library, nor literary society, not even the advantage of a resident clergyman'; consequently, he had none of those auxiliary helps and material assistances which we are accustomed to consider as indispensable to the acquisition of a sound practical education."

The village in question lies in the parish of Willerby, close to the sea coast, in the East Riding of Yorkshire. It is a few miles from Scarborough, which has grown within the last forty or fifty years from a small fishing town to a large and important watering place for tourists. Staxton was a small and straggling wayside village,

and the local shoemaker may have been a prominent personage in his way.

"The history of his youth, therefore, may be considered as forming another example of the noble illustrations which are upon record, of the successful 'pursuit of knowledge under difficulties.' Unfavorable and adverse as the circumstances were in which he was placed, yet he seems at a very early period to have had a glimmering presentiment or cheering presage that he would one day be a scholar and mathematician."

"At the age of thirteen he went to sea, and made a voyage to the West Indies; and, in scarcely three years afterwards, we find him aboard of a small vessel, where the captain and officers had been carried off by yellow fever, taking the command; and, from his knowledge of mathematics as applied to navigation, he brings the ship safely into port. At the age of sixteen he recommenced a vigorous course of self-training; at the same time he began to teach—the day being devoted to school duties, while his evenings were dedicated to the study of mathematics and general literature. Thus he was occupied for seven or eight long years, during which period he began to contribute to several mathematical journals and periodicals, and through them he ultimately became known to men of science."

Life in America.

After about twenty-five years in England, "where his life had been one continual struggle through years of trial, privation and toil," he came to New York, in 1830, doubtless in the company of his wife, because the date of his marriage is not known. It is stated that he at once began teaching in that city, and afterwards taught in Staten Island. "Soon after this he was appointed Professor of Mathematics and Natural Philosophy at Flushing Institute." This noteworthy school was under the care of a distinguished man already mentioned, the Rev. Dr. Muhlenberg, who expanded the school into a college, which took the name of St. Paul's. The college, after a course of some prosperity, and great earnestness among the students, amounting to positive enthusiasm, was finally given up, Muhlenberg entering on a life which was equally useful with that of a scholar, but which suited his genius better, that of a rector of a city church. In this work he continued till his death many years after. He was particularly noted for the catholicity of his sentiments. My own father, who was a New York minister

of another denomination, was a warm admirer of his, and never wavered in his strong affection for this large-hearted man. One thing that he did, among others, was the founding of St. Luke's Hospital. If he undertook any task, he entered into it with vigor and pertinacity, the deepest thing in his nature being earnestness. He was in fact by inheritance a kind of mystic. His biography, which I had not read until I looked it up in search of facts concerning Gill, and which does full justice to his memory, unfortunately does not mention Gill. The latter must have found his life at St. Paul's College entirely congenial. It had been, of course, a great change from a precarious existence among minor places in schools, and the companionship of such a man as Muhlenberg, and of the sort of men that he gathered about him, must have formed an oasis in his life. Flushing was a charming spot. College Point, as the site of the buildings came to be called, was a beautiful part of the village, sufficiently retired to be the home of scholars, and the main building was one of much architectural merit, though only built of wood. I remember it well, though I saw it only once, sixty years ago. Both Muhlenberg and Gill had gone elsewhere, the college having given place to an institute once more. In 1852 it had been bought by a friend of my father's, whose son's acquaintance was made by me that day, ending in a lifelong friendship. The boy became famous as R. W. Gilder, the poet and editor, whose youth was passed amid such surroundings.

The precise date when Gill joined the small body which formed the faculty of the college is not known, but from the fact given by Dr. Muhlenberg, that he had been employed on Staten Island, after work in a school in Westchester, it was probably as late as 1832. The next statement about Mr. Gill which has been repeatedly copied, is that "in 1836 appeared the first number of the *Mathematical Miscellany.*" This was a magazine which continued until 1839, and to which Gill gave an immense amount of hard work.

The Mathematical Miscellany.

The first step was taken in the new enterprise by the announced issue of a first number, preceded by a short "advertisement." This was as follows:

"This first number is an experiment. Its character and claims will be better explained by giving a specimen than a promise. The high estimate in which such works are held by European mathe-

maticians, and the fact of the great improvement in analysis, and that elegant problems in print appeared in this form are due to the discussions and investigations which they are calculated to bring forward," etc.,—in short, a laborious and diffuse attempt to explain something which was at first said to need no explanation. It went on to say that the names alone, if given, would be a guaranty of respectability on the part of those who have given assurance of support to the publication. Gill had, in fact, given much labor to the necessary preliminary correspondence, and there were certainly enough helpers in the field.

The main object of the publication was partly concealed until after the second number appeared. From this time forward special prominence was given to the presentation of problems, stating the time in which answers would be received, and then giving subsequently the names of those who answered, much as is now done in some publications devoted to young people. Emulation was plainly and forcibly appealed to. The time was ripe for such an enterprise. In this first number, it was announced that twenty new questions would be asked, which were accordingly stated on page 53. Answers would be accepted before August 1, 1836, and the next number would be issued on October 1, 1836. (The first number was apparently issued on March 1, 1836.)

The second number was obviously awaited with impatience by those interested. The following names are given among those of the contributors who are said to have answered the whole of the twenty questions satisfactorily: Professor Avery, of Hamilton College; Professor Catlin, of the same; Oren Root, of the same; Professor Theodore Strong, of Rutgers College. G. R. Perkins, of the Liberal Institute, Clinton, and others are mentioned as having answered some of the questions; and among the names given is the great name of B. Peirce, of Harvard University. There was also announced the promotion of a "Junior Department" composed of students only, which would in future add to the interest. Less abstruse questions were to be asked of the juniors. In the second number were given fifteen questions, to be answered by March 1, 1837. The third number would be issued two months later.

The third number contained, as answering all the fifteen questions, Avery, Catlin, Peirce, Root, Strong. These five could be relied upon, evidently. For the next number, the fourth, there were fifteen questions propounded, as well as in the third number,

to be answered by February 1, 1838. And the same five names appeared in the fourth number.

The same five names were repeated in the fifth number.

In the sixth number, answering all of the fifteen questions previously proposed, these names appeared: Avery, Catlin, Root, Strong. In the seventh number the names were Avery, Catlin, Peirce, Perkins, Strong.

The competition ended in the last (eighth) number of the "*Mathematical Miscellany,*" the same names appearing as in the seventh. No public notice was given of the stoppage.

In giving this brief summary of the chief gladiators in this contest for superiority, it must not be forgotten that many names appeared of those who answered some though not all of the questions. There were among those who were able to answer all of the questions, three of the five who are noted above as reliable. Of the five, Professor Peirce was one who did not continue throughout; and the other was a man of brilliant genius, Oren Root, of Clinton, N. Y. (father of the present United States Senator from his native state). Mr. Root, the father, was subsequently professor at the same college, Hamilton.

Returning to the second number, it contained a list of supporters which justified Gill's rosy anticipations. The two who gave their names for the largest amount were, "Rev. W. A. Muhlenberg, D.D., Principal of St. Paul's, $50 per annum," and "C. Gill, Professor of Mathematics, $20 per annum." The others follow:

L. Van Bokkelen, Mathematical Instructor	$10	per annum.
G. B. Docharty, Mathematical Instructor	10	" "
Professor C. Avery, Hamilton College, Clinton	15	" "
Professor M. Catlin, Hamilton College, Clinton	10	" "
O. Root, Mathematical Tutor, Hamilton College, Clinton	10	" "
G. R. Perkins, Liberal Institute, Clinton	5	" "
William Lenhart, Esq., York, Penna.	10	" "
J. F. Macully, Esq., New York	5	" "
Evans Hollis, Esq., Rye, N. Y.	5	" "

At the same time the price of subscription was fixed at $2 per annum.

It may be suspected that the subscription list itself was not very much larger. Professors Peirce and Strong are probably the only ones known to be on the list of subscribers in addition to the first contributors of money.

The names of Avery and Catlin, professors of mathematics at Hamilton, are on every list; and that of Theodore Strong completed the trio which was on each published statement of those who answered every question. Avery was perhaps the eldest of the triumvirate, which might have been called the invincible three. Catlin was equally certain to solve any problem which presented itself, and a sister of the third member of the three, Strong, is stated to have been married to Avery. I take this item of personal news from a pamphlet publication of the National Academy of Sciences, dated in 1879 from Washington: "A Memoir of Theodore Strong, LL.D.," by no less a person than that veteran ex-actuary, Joseph P. Bradley, of the Supreme Court of the United States.

Bradley's testimony to Strong's transcendent merit as a mathematician is very convincing. The history of his life is easily stated. Born in South Hadley, perhaps the same as the present Amherst, in 1790, he was graduated at Yale in 1812, where he had taken the prize in mathematics, and was at once chosen by President Backus, called by Bradley "a man of brilliant parts," as tutor in Hamilton College. He remained tutor for four years, followed by eleven others as professor of mathematics, retiring in 1827 to take the same chair in Rutgers College, New Brunswick, N. J. The journey there took four days. In 1859 he relinquished active work, to become in 1863 one of the earliest members of the National Academy of Sciences. He died in 1869, leaving a name practically unsurpassed in his part of this country, as is stated by Cajori in a note to his "History of Mathematics." Bradley's judgment is that "in his day he had hardly a peer in grasp and power of intellect," and that "his ability was so generally recognized at the period of his settlement in New Jersey that a prominent mathematician of New York, who, about that time, published a series of difficult problems which he challenged all the mathematicians in America to solve, expressly excepted Dr. Bowditch, Professor Strong, and one other." Bradley closes his memoir with a sentence which I am tempted to quote: "It is right that we should honor him in our memories and not neglect to speak his praise."

The invincible triumvirate of mathematicians furnished by Hamilton College are together so exceedingly noteworthy that I may say of Dr. Azel Backus, its president, that he had a good nose for youthful genius when choosing his staff of professors. Strong,

Avery and Catlin, followed by Root, formed a galaxy of unapproached talent.

Close of the First Paper.

I am so overwhelmed with the unexpected wealth of the material that I have gathered together that the space which I had expected to occupy, in placing before the society some record of the life and doings of Charles Gill, has been filled, and I must therefore defer the continuation of the story till the next meeting. The issue of the *Mathematical Miscellany* was only the beginning of his active work in America. The chief object of that work has already been indicated, although it was not clearly stated to the public. Many papers on mathematics will be mentioned, of which he himself was a large contributor, although the apparently ubiquitous personality of Theodore Strong may be responsible for much. (Judge Bradley counts twenty-two contributions of Strong's to the *Miscellany*, though only eight numbers in all were published.) In 1855 Gill's laborious and useful life was prematurely ended. I shall later quote from Muhlenberg's eulogy of his worth as a man, and I can sincerely say that my own judgment of his character is exceedingly high.

Mortality Surplus and Its Distribution.

BY

HENRY MOIR.

In 1907, the writer discussed this subject very briefly and compiled from the Gain and Loss Exhibits one or two tables showing the sources of the surplus earnings of *eight* representative companies. As those tables indicate the trend of events, I now submit the following figures in continuation—although in doing so I must convey a note of warning, as several of the companies have changed their reserve basis, and the figures must therefore be read with intelligent caution:

TABLE I.

Analysis of Gain & Loss Exhibits of Eight Companies.

Year Ending 31st Dec.	Profit from Loading.	Surplus Interest.	Profit from Mortality.	Gain from Surrenders and Lapses.	Profit or Loss on Investments.
1907	$17,840,378	$22,433,167	$16,866,149	$7,771,412	*$-79,326,844*
1908	17,501,371	25,296,804	18,144,949	6,431,989	72,977,452
1909	16,266,121	29,177,057	19,428,205	4,712,695	12,606,631
1910	16,140,828	32,506,383	20,342,828	4,672,824	5,255,661
1911	13,752,010	36,033,281	19,747,166	4,873,841	895,715
1912	14,337,039	38,982,291	22,168,548	5,417,454	*-12,259,351*
Totals 1907–1912 6 years.	95,837,747	184,428,983	116,697,845	33,880,215	149,264

Note.—The "Miscellaneous" gains and losses have been omitted from this table because large sums have been transferred to special reserves, contingency funds, etc., and the figures are misleading unless individually explained.

TABLE II.

Proportion of Profit from Principal Sources.

	Loading.	Interest.	Mortality.	Surrender.
1907	27.5%	34.5%	26.0%	12.0%
1908	26.0	38.0	27.0	9.0
1909	23.4	41.9	27.9	6.8
1910	22.0	44.1	27.6	6.3
1911	19.0	48.4	26.0	6.6
1912	17.7	48.2	27.4	6.7

This table if added across gives 100 per cent. each year, so that any abnormal change in one item affects all the other percentages for the same year.

These figures are interesting, especially when read with those given on page 220 of Vol. X of the *Transactions*—they bring out phases and tendencies in the evolution of our business. Despite the larger business of recent years, the profit from loading has been diminishing since 1907. This may be partly caused by the increased new business (through the heavy expense), but probably even more by the lower interest rates now in use for reserve purposes. Turning to surplus interest, the figures are most encouraging and an increase from about $22,500,000 to nearly $39,000,000 in six years looks eminently satisfactory. However, the decreased loading and the increased interest are both exaggerated by the changed valuation bases of the companies. In 1907 four of the eight selected companies were using 3½ per cent. interest for their new business, whereas three of these companies have since changed to 3 per cent., leaving only one of the eight companies now valuing its new business at 3½ per cent. Moreover, all of them have now a larger proportion of their total business on a lower valuation basis, whereby the loadings appear less, and the surplus interest correspondingly greater, while there has even been a tendency to change the interest to a lower rate on some of the older policies.

The gain from surrenders and lapses has taken the course we anticipated. There is a minimum below which this profit (if profit it can be called) will not fall so long as the companies carry full reserves for the first year. Since under present conditions 10 per cent. may be treated as a minimum lapse rate in the first year, the full reserve of this lapsed business will appear as "profit" in the Gain and Loss Exhibit, although it is no true profit, but may rather be viewed as a refund of initial expenditures.

But it is with the profit from mortality that I wish briefly to deal. This item has shown a normal development during recent years. The following table shows the mortality profit in the eight companies in relation to the outstanding sums at risk. No distinction is made as between new business and old; nevertheless the table is an interesting one and may convey some lessons to us.

TABLE III.

MORTALITY PROFIT PER $1,000 NET RISK OF EIGHT COMPANIES.

Year.	Total Sums Insured Outstanding (in Millions).	Total Reserve (in Millions).	Net Aggregate Risk (in Millions).	Mortality Profit (in Millions).	Mortality Profit per $1,000 at Risk. {(4) ÷ (3)} × 1000.
	(1)	(2)	(3)	(4)	(5)
1907	$6,648	$1,575	$5,073	$16.9	$3.3
1908	6,721	1,656	5,065	18.1	3.6
1909	6,882	1,744	5,138	19.4	3.8
1910	7,107	1,831	5,276	20.3	3.9
1911	7,389	1,922	5,467	19.7	3.6
1912	7,737	2,002	5,735	22.2	3.9

There have recently been published the rates of mortality as experienced by combined American companies—Medico-Actuarial Investigation, Vol. I, pp. 89–90. These rates were deduced from a record of policies, without taking into consideration the amounts insured or eliminating duplicates. It is almost safe to assume that the mortality rates by amounts will work out considerably higher than the rates by policies now made available. My present purpose, however, deals not so much with the exact mortality rates as with the trend of the mortality from age to age, and especially the relationship of the true mortality to the American Experience at young ages and at old. It is reasonable to assume that the amounts of insurance in individual policies at young ages—say, from 20 to 40 —are smaller than the amounts at the older ages, since relatively few men who are under forty take out large policies.* Arguing from theoretical grounds only, it would appear therefore as if a mortality table formed from amounts might be expected to show even higher mortality rates at the older ages than the table recently

* The above was written from *a priori* reasoning only, and without definite statistics to support the statement; but through the kindness of Mr. W. A. Hutcheson I have been supplied with an analysis of the policies issued by the Mutual Life in 1912 in force on Dec. 31st, and the following table shows the result:

Ages.	Number of Policies.	Average Amount of Each Policy.
21 to 30	22,605	$1,821
31 " 40	18,935	2,577
41 " 50	10,567	3,354
51 " 60	3,398	3,788

This table it will be observed amply confirms the attitude I have taken on this question.

published, and known as the M. A. Table. Be this as it may, the questions discussed and the methods used in this note can be applied to any new mortality experience which may become available, whether it be the individual experience of a particular company, or a combined experience which may afterwards be prepared, showing the loss ratio by amounts. If we follow the assumption underlying the select and ultimate method of valuation, viz.: that the additional saving from mortality on account of recent selection is applied during the first five years to meet the initial expenses, then any questions dealing with mortality surplus should deal with ultimate rates from the fifth year onwards. The following table gives some preliminary facts:

TABLE IV.

COMPARISON OF NEW MORTALITY RATES WITH AMERICAN TABLE.

Age.	American Exp. 1000 q_x.	New M. A. Table 1000 q_x.	Difference per $1,000.	Percentage American to M. A.
	(1)	(2)	(3)	(4)
25	8.1	4.7	$3.4	172.3%
30	8.4	4.9	3.5	171.4
35	8.9	5.1	3.8	174.5
40	9.8	5.7	4.1	171.9
45	11.2	7.5	3.7	149.3
50	13.8	10.6	3.2	130.2
55	18.6	15.8	2.8	117.7
60	26.7	24.0	2.7	111.2
65	40.1	39.0	1.1	102.8
70	62.0	61.7	.3	100.5
75	94.4	91.9	2.5	102.7
80	144.5	137.2	7.3	105.3

It will be observed that the ratio of the American Experience Table (now almost universally used in the United States) to the new M. A. Table is quite high at the younger ages—being over 170 per cent., up to age 40, and then the ratio decreases with wonderful regularity to age 69 when it is 100.4 per cent., and at age 70, 100.5 per cent. After age 70 it increases again, but we need not be surprised at this, since we know that the American Experience Table was arbitrarily adjusted at the older ages and the rates of mortality are abnormally high above age 80.

It is interesting to observe that the difference shown in column (3) of Table IV agrees roughly with column (5) of Table III, which takes into consideration the amount of each claim; but on the other hand Table III also includes as mortality profit the gain

from selection in the first five years on a large volume of new business. Apparently the two influences offset one another with the result that we have a striking similarity in the two tables, deduced as they are in such widely different ways.

If a company computes its premiums and makes its reserve valuations by the American Table 3 per cent., and experiences the mortality of the M. A. Table, we shall under ordinary life policies have the results which appear in the next table. In theory it is correct to distribute mortality profit, when dealt with separately, as a function of the cost of insurance on the reserve basis, and I understand that this theory is sometimes carried into practice. As such surplus in the aggregate often amounts to 20 per cent. or more of the cost of insurance, I have inserted in the table a column under each age of issue showing 20 per cent. of the cost of insurance by the American 3 per cent. table.

TABLE V.

PROFIT FROM MORTALITY ON ORDINARY LIFE POLICY OF $1,000.

Year.	Age 25 at Issue.			Age 30 at Issue.			Age 40 at Issue.			Age 50 at Issue.		
	Amt. at Risk.	Mort. Profit.	20% of Cost of Ins.	Amt. at Risk.	Mort. Profit.	20% of Cost of Ins.	Amt. at Risk.	Mort. Profit.	20% of Cost of Ins.	Amt. at Risk.	Mort. Profit.	20% of Cost of Ins.
5	$954	$3.3	$1.6	$944	$3.6	$1.7	$916	$3.7	$2.0	$877	$2.5	$3.1
10	901	3.4	1.6	880	3.6	1.7	823	2.7	2.2	749	2.1	3.7
15	840	3.4	1.6	806	3.1	1.7	722	2.0	2.5	622	.9	4.6
20	770	2.9	1.7	724	2.4	1.9	617	1.7	3.0	502	.1	5.7
25	691	2.3	1.8	635	1.8	2.2	512	.8	3.8	394	.9	6.9
30	606	1.7	2.1	542	1.5	2.7	413	.1	4.7	294	1.5	7.8
35	518	1.5	2.6	450	.7	3.3	324	.7	5.6	203	4.7	8.6
40	430	.6	3.2	363	.1	4.1	242	1.3	6.4	—	—	—

Similar figures can of course be made up very readily for all classes of policies, but there is no need to enlarge and amplify this note in such ways. The 20-payment life policy is, however, a very important one and it also introduces more of the investment element; accordingly the figures for this form are also given in Table VI, page 22.

One of the most striking features about these tables is the general tendency for the true mortality profit to decrease with the duration of the insurance, except at the very old ages, and at such ages the question is an unimportant one. Indeed, the surplus, as found by taking the difference in mortality between the two tables, is large

at the younger ages and quite small at the older ages; whereas the contrary holds true if we deal with the question theoretically and assume the surplus to be a percentage of the cost of insurance, the amount in that case being relatively small at young ages and quite large at the older ages. In the case of an ordinary life policy the surplus distributed as a percentage of the cost of insurance would increase consistently with the duration. It is only necessary to direct attention to this condition in order to show how grossly unfair such form of distribution may prove.

TABLE VI.

PROFIT FROM MORTALITY ON 20 PAYMENT LIFE POLICY OF $1,000.

Year.	Age 25 at Issue.			Age 30 at Issue.			Age 40 at Issue.			Age 50 at Issue.		
	Amt. at Risk.	Mort. Profit.	20% of Cost of Ins.	Amt. at Risk.	Mort. Profit.	20% of Cost of Ins.	Amt. at Risk.	Mort. Profit.	20% of Cost of Ins.	Amt. at Risk.	Mort. Profit.	20% of Cost of Ins.
5	$905	$3.1	$1.5	$894	$3.4	$1.6	$869	$3.3	$1.9	$839	$2.3	$2.9
10	791	3.0	1.4	769	3.2	1.5	717	2.4	1.9	662	1.9	3.3
15	656	2.7	1.3	621	2.4	1.3	540	1.5	1.9	463	.7	3.4
20	495	1.9	1.1	445	1.5	1.2	333	.9	1.6	223	.0	2.5
25	445	1.5	1.2	390	1.1	1.4	277	.4	2.1	175	.4	3.1
30	390	1.1	1.4	333	.9	1.7	223	.0	2.6	131	.7	3.5
35	333	.9	1.7	277	.4	2.1	175	.4	3.1	90	2.1	3.9
40	277	.5	2.1	223	.0	2.6	131	.7	3.5	—	—	—

Occasionally the distribution of mortality surplus has been made by the use of a decreasing series of percentages adopted in a more or less arbitrary way. I have seen a method in use whereby 40 per cent. of the cost of insurance was given at ages 25 and under, and the series decreased by 1 per cent. per annum until a small percentage was reached, which was left constant for life thereafter. This system was applied irrespective of the age at issue, and the following table shows that although it represented only the individual judgment of the actuary in charge and was without scientific foundation, nevertheless the results were more equitable, when measured by our new standards, than the percentage of the cost of insurance uniform at all ages. Nevertheless, the arbitrary method, while giving figures which are "better" by comparison, can scarcely be said to attain to a satisfactory standard. It is sufficient to give these results for the ordinary life policy only (American 3 per cent.); indeed the point is mentioned chiefly to show that actuaries

have been anticipating some such condition as is now brought to light.

TABLE VII.

MORTALITY ALLOTMENT BY ARBITRARY GRADATION—ORDINARY LIFE.

Year.	Age 25 at Issue.		Age 30 at Issue.		Age 40 at Issue.		Age 50 at Issue.	
	Percentage of Cost.	Amt. per $1000.	Percentage of Cost.	Amt. per $1000.	Percentage of Cost.	Amt. per $1000.	Percentage of Cost.	Amt. per $1000.
5	35%	$2.79	30%	$2.50	20%	$1.98	10%	$1.53
10	30%	2.39	25%	2.11	15%	1.62	5%	.93
15	25%	2.01	20%	1.75	10%	1.26	5%	1.14
20	20%	1.67	15%	1.42	5%	.76	5%	1.42
25	15%	1.35	10%	1.10	5%	.94	5%	1.72
30	10%	1.05	5%	.67	5%	1.17	5%	1.94
35	5%	.64	5%	.83	5%	1.41	5%	2.15
40	5%	.80	5%	1.03	5%	1.59	5%	2.44

Some months before I had prepared these notes or had thought of this method of analysis I was talking with Mr. Robert Henderson on the general subject of distribution of mortality surplus, when he expressed himself as being of the opinion that probably a *constant* function of the amount at risk would be more just than a *percentage* function of the cost of insurance. The foregoing tables indicate that there is much to commend this view, which would bring out an allotment as follows:

TABLE VIII.

MORTALITY ALLOTMENT BY CONSTANT ($3 PER $1,000) ON THE COST OF INSURANCE—ORDINARY LIFE.

Year.	Age 25 at Issue.		Age 30 at Issue.		Age 40 at Issue.		Age 50 at Issue.	
	Amount at Risk.	Surplus Allotment.	Amount at Risk.	Surplus Allotment.	Amount at Risk.	Surplus Allotment.	Amount at Risk.	Surplus Allotment.
5	$954	$2.9	$944	$2.8	$916	$2.7	$877	$2.6
10	901	2.7	880	2.6	823	2.5	749	2.2
15	840	2.5	806	2.4	722	2.2	622	1.9
20	770	2.3	724	2.2	617	1.9	502	1.5
25	691	2.1	635	1.9	512	1.5	394	1.2
30	606	1.8	542	1.6	413	1.2	294	.9
35	518	1.6	450	1.4	324	1.0	203	.6

For purposes of general equity, as dealing with the mortality function only, it will be observed that this suggestion (which so far as the writer knows is not yet in actual use) would prove much more equitable than the percentage method—as will be readily seen by

comparing the grade of the above figures with those brought out in Table V.

It has been the custom of some companies to ignore altogether the mortality as a factor in distribution. For example, several companies use the "two-factor" method of applying the contribution plan—namely: (1) surplus interest, and (2) all other surplus; dividing the first according to reserve values, and the second according to loadings. It, therefore, becomes of interest to observe to what extent a percentage of the loadings may agree with the mortality surplus as derived in Table IV. To illustrate this it is necessary to use actual premiums and loadings and we can reasonably assume the American Experience 3 per cent. loaded 12½ per cent. and 12½ per cent., giving the following figures in use by a good many companies:

TABLE IX.

40 PER CENT. OF LOADINGS PER $1,000.

Age.	Ordinary Life $1,000.		20 Payment Life $1,000.	
	Premium.	Loading.	Premium.	Loading.
25	$20.14	$4.03	$30.12	$5.14
30	22.85	4.57	32.87	5.68
40	30.94	6.19	40.38	7.24
50	45.45	9.09	52.87	9.92

On the basis that 40 per cent. of the loading might represent a fair refund on account of mortality savings, we have results which, of course, are constant for all durations.

	Ordinary Life				20 Payment Life			
	Age 25	30	40	50	Age 25	30	40	50
Constant for all Durations.	$1.6	$1.8	$2.5	$3.6	$2.1	$2.3	$2.9	$4.0
					Ceasing after 20 years.			

But when we enter on this phase of the subject, we are introducing other elements which affect the problem, as, for example: (1) the general custom in this country of loading ordinary life premiums as a percentage of the net premium and carrying the same principle into other classes; (2) the custom of paying commissions as a percentage of the premium, making the initial expense of a policy at an old age higher than at younger ages; and (3) the effect

of actual expenditures incurred, especially the fact that expenses diminish as the duration increases, since commissions die out and collection becomes easier.

One of the most valid criticisms in theory of the "two-factor" method of distribution is that limited payment policies do not get any share of the mortality surplus after the policies become paid-up; but it is apparent from Table VI that this criticism is not of much consequence if the true mortality runs in accordance with the M. A. Table, since 20 payment policies, which form the great majority of such contracts, contribute but little after the twenty years run out. If a liberal interest factor is used in distribution the question becomes one of very slight importance, since paid-up policies have large reserves and the interest profit is therefore the principal element. Then again, under limited payment policies in this country a large number of them discontinue at the end of the premium paying period, and there is probably a selection exercised at that time against the companies; this has been found to be a notable feature in the case of "post-tontine" policies. All these considerations indicate that the criticism, while just in theory, has very little weight in practical equity.

This note is intended to be suggestive rather than conclusive, since reasonable equity in surplus distribution is attained by different companies in widely varying ways, and any dogmatic attitude on the subject is peculiarly out of place. Yet on giving careful consideration to the foregoing tables, to the premium loadings in general use which are relatively much more at older entry ages, to the reduced commissions and expenses after policies have been ten or more years in force, and to the varying mortality in different classes of policies, I am strongly inclined to the belief that we secure a more equitable distribution by merging the mortality surplus in the general earnings of the company and making the allotment in proportion to the loadings, than by using the mortality as a factor of the cost of insurance under our present reserve bases. The mortality profit is here shown to be, in general, a decreasing factor with the increase in duration or in age at issue, whereas, on the contrary, the savings from loadings increase with duration, since commissions cease and collections become automatic, while the savings from loadings also increase with the age at entry on account of the method of loading commonly used.

These forces, therefore, acting in opposite directions, can, in the

writer's opinion, be best merged and offset one against the other as is done in distributing general earnings as a percentage of the loadings.

The method of analysis herein set forth is a new one, so far as the writer is aware; but it is so direct and simple that it will doubtless appeal to the actuaries of companies, especially as it can be used with the individual figures of any company, or combination of companies, and can be applied to sections of the business when the relative statistics can be secured. It gives us facts and demonstrations in place of our nebulous impressions.

Concerning the American Experience Table of Mortality.

Second Paper.

BY

S. A. JOFFE.

It will be recalled that in my first paper* I advanced the hypothesis that in constructing the American Experience Table, Homans followed a method similar to that which Gill had adopted in constructing his mortality table.† My assumption was, that after having modified in a certain manner the curve which represented the reciprocals of the "adjusted rate of mortality" of the Mutual Life's fifteen years' experience (1843–1858), Homans had read off from that curve the values of his fundamental function $r_x (= l_x/d_x)$ at a number of equidistant points, had taken the reciprocals of those values for his q_x at the same equidistant points, and computed the intermediate values of q_x by finite difference interpolation. Starting then with $l_{10} = 100{,}000$, he computed the successive values d_x and l_{x+1}, by multiplying l_x by q_x (or, perhaps, dividing by r_x), and by subtracting d_x from l_x respectively.

The paper was discussed by Mr. D. P. Fackler at the Society's meeting of May, 1912,‡ and in the discussion he referred to a table of "the original basic rates of mortality as adjusted by Mr. Homans," which is appended to Mr. Fackler's "Remarks" before the National Insurance Convention at its first session in 1871.§ A part of this table contains a "beautifully graduated rate of mortality of the (American Experience) Table when starting with 10,000,000 persons, at age 10," accompanied by the following footnote: "This is the rate of mortality on which Mr. Homans wished the American Experience Table to be based." In my eyes, this table, coming from Homans himself, appeared to deserve careful study, despite the fact that I almost despaired of ever being able to establish the connection between this and the American Experience Table, in view of Mr.

* T. A. S. A., XII, 253–260. † l. c., XIII, 80–87.

‡ l. c., XIII, 159.

§ Official Report of National Insurance Convention (1871), pages 179–180.

Fackler's remark* that "There are such differences in the successive results obtained from it (the American Experience Table) that one is often led to think that he (Homans) has made some considerable error." However, as will appear from the following pages, this table and Mr. Fackler's suggestion,† that in order that its graduation should be perfect the table should be "made on a basis of ten millions of lives," have furnished me with the most important of the missing links in my hypothesis, and I consider it my pleasant duty to acknowledge that whatever success I have had in this investigation is due to Mr. D. P. Fackler's valuable suggestions.

The following is the part of the table just mentioned, in which I have changed the "Mortality per 1,000" to the customary rate of mortality per unit:

"Beautifully Graduated Rate of Mortality."

Age	Rate of Mortality.	Age	Rate of Mortality.	Age	Rate of Mortality.	Age	Rate of Mortality.
10	.007491	30	.008424	50	.013776	70	.061996
11	.007517	31	.008513	51	.014537	71	.067659
12	.007545	32	.008609	52	.015388	72	.073730
13	.007573	33	.008713	53	.016338	73	.080193
14	.007601	34	.008827	54	.017397	74	.087040
						75	.094366
15	.007630	35	.008951	55	.018575	76	.102323
16	.007661	36	.009088	56	.019884	77	.111074
17	.007694	37	.009240	57	.021333	78	.120817
18	.007729	38	.009406	58	.022937	79	.131752
19	.007768	39	.009589	59	.024715	80	.144446
						81	.158629
20	.007810	40	.009789	60	.026691	82	.174338
21	.007854	41	.010008	61	.028878	83	.191571
22	.007902	42	.010249	62	.031286	84	.211372
23	.007953	43	.010519	63	.033942	85	.235627
24	.008008	44	.010823	64	.036879	86	.265675
						87	.302939
25	.008066	45	.011166	65	.040124	88	.346861
26	.008128	46	.011557	66	.043703	89	.395883
27	.008195	47	.012003	67	.047651	90	.454133
28	.008266	48	.012515	68	.051999	91	.533333
29	.008342	49	.013104	69	.056773	92	.632911
						93	.732601
						94	.869565
						95	1.000000

With my hypothesis ever present in my mind, I first attempted to replace the values of q_x, for $x =$ multiple of 5, by their reciprocals. While in Gill's computations the values of r_x, for $x =$ multiple of

* Official Report of National Insurance Convention (1871), p. 179.

† l. c., p. 179.

5, were read off from the curve, correct to one decimal, the corresponding values of r_x in Homans' Table No. 1 were taken to two decimals, and I therefore decided to take the reciprocals, in the present case, also to two decimals. The rate of mortality q_{10} being .007491, its reciprocal $r_{10} = 133.49$. However, even in this first instance, I could not be certain that this was the correct value for the simple reason that my object was not to obtain the reciprocal of .007491, but to find the number whose reciprocal would be .007491, and the equation $1/r = .007491$ is indeterminate in the sense that it is satisfied by both $r = 133.49$ and $r = 133.50$. But it required very little ingenuity to surmise that in selecting a value for his initial r_{10}, Homans would have taken the round number $133\frac{1}{2}$ rather than 133.49.

In the case of age 15, the equation $1/r = .007630$ has again two solutions: $r = 131.06$ and $r = 131.07$, but, unfortunately, it is impossible to apply here the same analytical method as in the case of age 10, since one cannot tell *à priori* which of these two numbers was used by Homans. While at age 20 there is only one number (128.04) of which .007810 is the reciprocal, at age 25 there is an uncertainty again, whether to take 123.97 or 123.98 for the number r_{25} of which the reciprocal is $q_{25} = .008066$. These few cases, for which the results could not be determined with certainty, were sufficient to make me abandon this plan of attack and turn to the next step in my investigation—between the assumed values of q_x at the ages $x =$ multiple of 5, to interpolate the intermediate values of q_x at the other ages.

Commencing with the set of values: q_{10}, q_{15}, q_{20}, q_{25}, q_{30}, and q_{35}, I obtained by interpolation the intermediate values of q_x, which agreed with Homans' values so closely that better results could hardly be expected. Results almost as good, up to age 35, were obtained from a similar set of values beginning with age 20. But notwithstanding many trials with different sets no satisfactory results were obtained for ages above 35. It then occurred to me, before continuing this experimentation which had consumed a great deal of time without apparent hope of success, to turn my attention to the last stage of the investigation and to ascertain whether, having assumed all the values of Homans' auxiliary q'_x, it would not be possible to reproduce his principal q_x as given in the American Experience Table. As will be seen from the details of the table at the end of this paper, I succeeded in reproducing q_x

of the American Table for every age from 10 to 95, without any exception.

This table is sufficiently detailed to be in need of lengthy explanations. Column (6) contains the values of log d'_x found from the formula $\log d'_x = \log l'_x + \log q'_x$; column (7) gives the antilogarithms of (6), and in column (8) are entered the results of division of column (7) by 100, taken to the nearest unit. It will be seen that this column (8) of d_x is precisely the same as appears in the established form of the American Experience Table, and columns (9) and (10) of l_x and q_x are, of course, also those of the American Experience Table. In view of this complete agreement between the results, I believe, it is safe to assume that Homans followed the above method in deriving his figures for the American Experience Table. And it appears to me that until the values of q_x of the American Experience Table are reproduced just as perfectly by another method, which, for some reason, would appear to be more plausible, or until some one proves the present method to be defective, we shall be justified in believing that Homans not only could, but actually did, use this very method in his computations. For this reason I have thought it advisable to present here all the calculations in full detail.

Inasmuch as the connection between the auxiliary values q'_x and the principal values q_x has been satisfactorily established, we may now revert to the formation of the auxiliary values themselves. As stated before, the latter may be obtained for all the ages 10 to 34 inclusive by one continuous process from the set of quinary values of q'_x beginning with age 10. Applying the same formulas and arranging the details in the same manner as in my paper on Gill's Mortality Table,* we have the following results, where it will be noticed that of twenty values computed by interpolation there are only two instances in which the discrepancy between columns (2) and (3) is more than a unit:

Age.	10^6 Auxiliary q_x'	Leading Differences.	
		For 5-Unit Interval.	For 1-Unit Interval.
10	7,491	$\Delta = 139$	$\delta = 27.7792$
15	7,630	$\Delta^2 = 41$	$\delta^2 = -.7664$
20	7,810	$\Delta^3 = 35$	$\delta^3 = .8464$
25	8,066	$\Delta^4 = -9$	$\delta^4 = -.1424$
30	8,424	$\Delta^5 = 50$	$\delta^5 = .016$
35	8,951		

* T. A. S. A., XIII, 82–83.

VALUES OF $10^6 q_x'$ FOR AGES 10 TO 34.

(1) Age.	(2) Joffe's Value.	(3) Homans' Value.	(4) (2)-(3) to 1 Decimal.
10	7,491.	7,491	
11	7,518.7792	7,517	1.7
12	7,545.7920	7,545	.7
13	7,572.8848	7,573	—.1
14	7,600.7616	7,601	—.2
15	7,630.	7,630	
16	7,661.0672	7,661	.0
17	7,694.3360	7,694	.3
18	7,730.1008	7,729	1.1
19	7,768.5936	7,768	.5
20	7,810.	7,810	
21	7,854.4752	7,854	.4
22	7,902.1600	7,902	.1
23	7,953.1968	7,953	.1
24	8,007.7456	8,008	—.2
25	8,066.	8,066	
26	8,128.2032	8,128	.2
27	8,194.6640	8,195	—.3
28	8,265.7728	8,266	—.2
29	8,342.0176	8,342	.0
30	8,424.	8,424	
31	8,512.4512	8,513	—.5
32	8,608.2480	8,609	—.7
33	8,712.4288	8,713	—.5
34	8,826.2096	8,827	—.7

If we make our computations from the set beginning with age 20, equally satisfactory results are obtained only as far as age 35; after that the divergence between the computed values and those of Homans rapidly increases. Although several small sections of some 10 ages each have been reproduced by me from various sets in a palpably satisfactory manner, I am rather inclined to believe that the ages above 35 require some method which I have so far not discovered, and I shall consider the whole section of ages 35–70 as having baffled my efforts in that direction.

From age 70 to the end of the Table there remain 25 ages, and if up to this point I should have failed to show that the function q_x' was derived from a preliminary function r_x, I hope to adduce the most conclusive proof for this contention from the last 25 ages of the Table. For this purpose I shall write down the following 25

numbers, with their reciprocals to 7 decimal places, for a comparison of these reciprocals with the values of q'_x of the "beautifully graduated rate," for ages 70 to 94 inclusive:

Number.	Reciprocal.	Rate of Mortality.	Number.	Reciprocal.	Rate of Mortality.
16.130	.0619963	.061996	5.220	.1915709	.191571
14.780	.0676590	.067659	4.731	.2113718	.211372
13.563	.0737300	.073730	4.244	.2356268	.235627
12.470	.0801925	.080193	3.764	.2656748	.265675
11.489	.0870398	.087040	3.301	.3029385	.302939
10.597	.0943663	.094366	2.883	.3468609	.346861
9.773	.1023227	.102323	2.526	.3958828	.395883
9.003	.1110741	.111074	2.202	.4541326	.454133
8.277	.1208167	.120817	1.875	.5333333	.533333
7.590	.1317523	.131752	1.580	.6329114	.632911
6.923	.1444460	.144446	1.365	.7326007	.732601
6.304	.1586294	.158629	1.150	.8695652	.869565
5.736	.1743375	.174338			

I do not think that a comparison of the last two columns of the preceding table will leave any room for doubt that the values of q'_x for the 25 ages, 70 to 94, were actually taken by Homans as reciprocals of the above 25 numbers. It is especially difficult to conceive that such a series of numbers to 6 decimal places as, for instance, the last five in the table, namely .454133, .533333, .632911, .732601 and .869565, could have been obtained by any uniform method of computation except as reciprocals of 2.202, 1.875, 1.580, 1.365 and 1.150, which—it will be observed— have only 3 decimal places.

However encouraging, on the one hand, these results may be as verifying my earlier assumptions in general, they are, on the other hand, disappointing to me, because the succession of the five numbers just mentioned, 2.202, etc., upsets the theory which I had formed on this particular point before. From previous study I had concluded* that for ages 90 to 95, r_x was an exponential function of $95 - x$; but it is easily seen that the five numbers just quoted do not follow this law, representing as they do a very irregular series—so irregular, indeed, that I am almost tempted to say with Mr. Fackler that in this instance Homans did make some considerable error. It may not be out of place to give here the following details which brought me to my previous conclusions:

* T. A. S. A., XIII, 161.

For the last six ages of the American Experience Table we may write:

Age.	$r_x = \frac{l_x}{d_x}$.
90	2.200
91	1.878
92	1.577
93	1.362
94	1.167
95	1.

The fact that $r_{94} = 1.167 = 7/6$, and $r_{93} = 1.362 = (7/6)^2$ suggested that possibly r_{92}, r_{91} and r_{90}, as well, were successive powers of 7/6, or that, in other words, the values of r_x for the four ages, 91, 92, 93 and 94 formed geometric means between 1 and 2.2. From the equation $r_{94}^5 = 2.2$, by means of $\log 2.2 = .3424227$, we find the following values, to two decimal places:

$r_{94} = 1.17$, $r_{93} = r_{94}^2 = 1.37$, $r_{92} = r_{94}^3 = 1.60$, $r_{91} = r_{94}^4 = 1.88$, and $r_{90} = r_{94}^5 = 2.20$.

Now, starting with $l_{90} = 847$, we obtain successively:

$$d_{90} = \frac{847}{2.20} = 385, \quad l_{91} = 462,$$

$$d_{91} = \frac{462}{1.88} = 246, \quad l_{92} = 216,$$

$$d_{92} = \frac{216}{1.60} = 135, \quad l_{93} = 81,$$

$$d_{93} = \frac{81}{1.37} = 59, \quad l_{94} = 22,$$

$$d_{94} = \frac{22}{1.17} = 19, \quad l_{95} = 3.$$

Thus the values of l_x, for $x = 91, \ldots 95$, obtained by me, are 462, 216, 81, 22, and 3, which almost coincide with Homans' values: 462, 216, 79, 21, and 3. Notwithstanding, however, this striking coincidence in the results, I incline (for reasons already stated) towards the irregular series: 2.202, 1.875, etc., as the fundamental

series originally used by Homans, and I accept it as such with reluctance, because I have so far failed in discovering any plausible method by which he could have derived that series.

Finally, when we glance over the series of values of the preliminary function r_x, such as

Age.	r_x	Age	r_x
10	133.50	..	....
15	131.06 (or 131.07?)	70	16.130
20	128.04	71	14.780
25	123.97 (or 123.98?)	72	13.563
30	118.71	..	....
35	111.72	..	....

the question arises whether in using the graphic method as a first step in graduating his table, Homans could have read off his ordinates r_x correctly to two decimal places, and I am not ready to submit an answer to this question. It is possible, of course, that for his preliminary values he had taken such round numbers as $133\frac{1}{2}$, 131, 128, 124, $118\frac{3}{4}$, $111\frac{3}{4}$, and then modified these by some refining analytical process. For the end of the table, if the last 25 values were also obtained by the graphic method, there is little doubt that the fractions could not be read off to three decimal places without having that part of the curve drawn separately on a different, greatly enlarged scale, and even then the final values must have undergone some further refinement by an analytical method.

RÉSUMÉ.

In order to furnish a clear and succinct view of the various results as a whole, it may be stated by way of summary that the facts gathered so far make the writer believe that the construction by Homans of the American Experience Table consisted of the following three main operations:

1. The irregular curve of "General Experience 15 Years" in Diagram No. 4 facing page 24 of the Report on the Mutual Life's mortality and representing the "Numbers out of which one person will die in each year," was "smoothed out" by free hand drawing into a uniform curve having for ordinates 133.5 at age 10 and 16.13 at age 70. It is not certain whether the curve was extended to the point having for its ordinate 1 at age 95.

2. The ordinates r_x were read off at certain points, their recipro-

cals taken for q'_x at those points, and the intermediate values of q'_x computed by interpolation. The 85 ages of the Table fall into three groups: two, of 25 ages each, at the extremes, and one, of 35 ages, in the middle. In the first group: 10 to 35, all the q'_xs are derived by interpolation from one set with quinary intervals. The middle group, 35 to 70, remains so far without verification. In the last group, 70 to 95, each q'_x is a reciprocal of the corresponding value of r_x taken to three decimals; the process by which the latter values were determined has not been ascertained.

3. With a basis of $l'_{10} = 10{,}000{,}000$, the logarithms of d'_x were found from the formula $\log d'_x = \log l'_x + \log q'_x$, and the result of division of d'_x by 100, to the nearest unit, was taken for d_x, the "number dying," as it appears in the American Experience Table. With a basis of $l_{10} = 100{,}000$, the "number living" l_x was then computed from the formula $l_{x+1} = l_x - d_x$, and the final division of d_x by l_x produced the rate of mortality q_x of the American Experience Table.

CONCLUSION.

If my paper on Gill's Mortality Table has not as yet induced any member of this Society to undertake the verification of Gill's figures, so as to either supplement or rectify my work, it may be explained by the fact that Gill's Table is looked upon as having now a historical interest only (although this is an erroneous view, since it is becoming more and more apparent that Gill's methods were in a great measure utilized by Homans in his work subsequently). But in the present instance we are concerned with a Table that was introduced by the Mutual Life over half a century ago, and has since become one of the most important tables of modern times. It would probably be beyond human powers to calculate the number of hours that have been collectively expended by insurance companies in preparing different tables of premiums and of reserves from the American Experience Table, and in computing the premium rates charged and the values granted on the individual policies issued on the basis of that Table. For this alone, if for no other reason, the American Experience Table deserves on the part of the Actuarial Society of America that its members should endeavor to trace every figure in that Table to its original source, and I hope that the few results presented here will prove of sufficient interest to others to make them enter the same field of investigation which appears now to give promise of ultimate success.

DERIVATION OF PRINCIPAL q_x FROM AUXILIARY q_x'.

(1) Age.	(2) l_x', on basis of $l_{10}'=10,000,000$ and $l'_{x+1}=l_x'-d_x'$.	(3) log. l_x'	(4) Auxiliary q_x'.	(5) log. q_x'	(6) log. d_x' (3)+(5)	(7) d_x'	(8) $d_x=d_x'/100$ to nearest unit.	(9) l_x, on basis of $l_{10}=100,000$ and $l_{x+1}=l_x-d_x$.	(10) Principal q_x (8)÷(9).
10	10,000,000	7.0000000	.007491	$\bar{3}$.8745398	4.8745398	74,910	749	100,000	.007490
11	9,925,090	6.9967345	.007517	$\bar{3}$.8760446	4.8727791	74,607	746	99,251	.007516
12	9,850,483	6.9934575	.007545	$\bar{3}$.8776592	4.8711167	74,322	743	98,505	.007543
13	9,776,161	6.9901684	.007573	$\bar{3}$.8792680	4.8694364	74,035	740	97,762	.007569
14	9,702,126	6.9868669	.007601	$\bar{3}$.8808707	4.8677376	73,746	737	97,022	.007596
15	9,628,380	6.9835532	.007630	$\bar{3}$.8825245	4.8660777	73,465	735	96,285	.007634
16	9,554,915	6.9802268	.007661	$\bar{3}$.8842855	4.8645123	73,200	732	95,550	.007661
17	9,481,715	6.9768869	.007694	$\bar{3}$.8861522	4.8630391	72,952	729	94,818	.007688
18	9,408,763	6.9735325	.007729	$\bar{3}$.8881233	4.8616558	72,720	727	94,089	.007727
19	9,336,043	6.9701628	.007768	$\bar{3}$.8903092	4.8604720	72,522	725	93,362	.007765
20	9,263,521	6.9667761	.007810	$\bar{3}$.8926510	4.8594271	72,348	723	92,637	.007805
21	9,191,173	6.9633709	.007854	$\bar{3}$.8950909	4.8584618	72,187	722	91,914	.007855
22	9,118,986	6.9599466	.007902	$\bar{3}$.8977370	4.8576836	72,058	721	91,192	.007906
23	9,046,928	6.9565012	.007953	$\bar{3}$.9005310	4.8570322	71,950	720	90,471	.007958
24	8,974,978	6.9530334	.008008	$\bar{3}$.9035241	4.8565575	71,872	719	89,751	.008011
25	8,903,106	6.9495416	.008066	$\bar{3}$.9066582	4.8561998	71,812	718	89,032	.008065
26	8,831,294	6.9460243	.008128	$\bar{3}$.9099837	4.8560080	71,781	718	88,314	.008130
27	8,759,513	6.9424800	.008195	$\bar{3}$.9135490	4.8560290	71,784	718	87,596	.008197
28	8,687,729	6.9389063	.008266	$\bar{3}$.9172954	4.8562017	71,813	718	86,878	.008264
29	8,615,916	6.9353014	.008342	$\bar{3}$.9212702	4.8565716	71,874	719	86,160	.008345
30	8,544,042	6.9316633	.008424	$\bar{3}$.9255184	4.8571817	71,975	720	85,441	.008427
31	8,472,067	6.9279894	.008513	$\bar{3}$.9300826	4.8580720	72,123	721	84,721	.008510
32	8,399,944	6.9242764	.008609	$\bar{3}$.9349527	4.8592291	72,315	723	84,000	.008607
33	8,327,629	6.9205214	.008713	$\bar{3}$.9401677	4.8606891	72,559	726	83,277	.008718
34	8,255,070	6.9167207	.008827	$\bar{3}$.9458131	4.8625338	72,867	729	82,551	.008831
35	8,182,203	6.9128703	.008951	$\bar{3}$.9518716	4.8647419	73,239	732	81,822	.008946
36	8,108,964	6.9089653	.009088	$\bar{3}$.9584683	4.8674336	73,694	737	81,090	.009089
37	8,035,270	6.9050005	.009240	$\bar{3}$.9656720	4.8706725	74,246	742	80,353	.009234
38	7,961,024	6.9009689	.009406	$\bar{3}$.9734050	4.8743739	74,881	749	79,611	.009408
39	7,886,143	6.8968647	.009589	$\bar{3}$.9817733	4.8786380	75,620	756	78,862	.009586
40	7,810,523	6.8926801	.009789	$\bar{3}$.9907383	4.8834184	76,457	765	78,106	.009794
41	7,734,066	6.8884079	.010008	$\bar{2}$.0003473	4.8887552	77,403	774	77,341	.010008
42	7,656,663	6.8840396	.010249	$\bar{2}$.0106815	4.8947211	78,473	785	76,567	.010252
43	7,578,190	6.8795655	.010519	$\bar{2}$.0219745	4.9015400	79,715	797	75,782	.010517
44	7,498,475	6.8749730	.010823	$\bar{2}$.0343477	4.9093207	81,156	812	74,985	.010829
45	7,417,319	6.8702469	.011166	$\bar{2}$.0478976	4.9181445	82,822	828	74,173	.011163
46	7,334,497	6.8653703	.011557	$\bar{2}$.0628451	4.9282154	84,765	848	73,345	.011562
47	7,249,732	6.8603219	.012003	$\bar{2}$.0792898	4.9396117	87,019	870	72,497	.012000
48	7,162,713	6.8550776	.012515	$\bar{2}$.0974309	4.9525085	89,641	896	71,627	.012509
49	7,073,072	6.8496081	.013104	$\bar{2}$.1174039	4.9670120	92,686	927	70,731	.013106
50	6,980,386	6.8438795	.013776	$\bar{2}$.1391231	4.9830026	96,162	962	69,804	.013781
51	6,884,224	6.8378551	.014537	$\bar{2}$.1624748	5.0003299	100,076	1,001	68,842	.014541
52	6,784,148	6.8314953	.015388	$\bar{2}$.1871822	5.0186775	104,394	1,044	67,841	.015389
53	6,679,754	6.8247606	.016338	$\bar{2}$.2131989	5.0379595	109,134	1,091	66,797	.016333
54	6,570,620	6.8176063	.017397	$\bar{2}$.2404744	5.0580807	114,309	1,143	65,706	.017396

DERIVATION OF PRINCIPAL q_x FROM AUXILIARY q_x' (*Continued*).

(1) Age.	(2) l_x', on basis of $l_{10}=10,000,000$ and $l'_{x+1}=l_x'-d_x'$.	(3) log. l_x'	(4) Auxiliary q_x'.	(5) log. q_x'	(6) log. d_x' (3)+(5)	(7) d_x'	(8) $d_x=d_x'/100$ to nearest unit.	(9) l_x, on basis of $l_{10}=100,000$ and $l_{x+1}=l_x-d_x$.	(10) Principal q_x (8)÷(9).
55	6,456,311	6.8099845	.018575	$\bar{2}$.2689288	5.0789133	119,926	1,199	64,563	.018571
56	6,336,385	6.8018416	.019884	$\bar{2}$.2985038	5.1003454	125,993	1,260	63,364	.019885
57	6,210,392	6.7931190	.021333	$\bar{2}$.3290519	5.1221709	132,486	1,325	62,104	.021335
58	6,077,906	6.7837540	.022937	$\bar{2}$.3605366	5.1442906	139,409	1,394	60,779	.022936
59	5,938,497	6.7736766	.024715	$\bar{2}$.3929606	5.1666372	146,770	1,468	59,385	.024720
60	5,791,727	6.7628081	.026691	$\bar{2}$.4263648	5.1891729	154,587	1,546	57,917	.026693
61	5,637,140	6.7510588	.028878	$\bar{2}$.4605671	5.2116259	162,789	1,628	56,371	.028880
62	5,474,351	6.7383327	.031286	$\bar{2}$.4953500	5.2336827	171,271	1,713	54,743	.031292
63	5,303,080	6.7245282	.033942	$\bar{2}$.5307374	5.2552656	179,997	1,800	53,030	.033943
64	5,123,083	6.7095314	.036879	$\bar{2}$.5667791	5.2763105	188,934	1,889	51,230	.036873
65	4,934,149	6.6932122	.040124	$\bar{2}$.6034042	5.2966164	197,978	1,980	49,341	.040129
66	4,736,171	6.6754274	.043703	$\bar{2}$.6405113	5.3159387	206,985	2,070	47,361	.043707
67	4,529,186	6.6560202	.047651	$\bar{2}$.6780720	5.3340922	215,820	2,158	45,291	.047647
68	4,313,366	6.6348163	.051999	$\bar{2}$.7159950	5.3508113	224,291	2,243	43,133	.052002
69	4,089,075	6.6116250	.056773	$\bar{2}$.7541418	5.3657668	232,149	2,321	40,890	.056762
70	3,856,926	6.5862413	.061996	$\bar{2}$.7923637	5.3786050	239,114	2,391	38,569	.061993
71	3,617,812	6.5584460	.067659	$\bar{2}$.8303256	5.3887716	244,778	2,448	36,178	.067665
72	3,373,034	6.5280207	.073730	$\bar{2}$.8676442	5.3956649	248,694	2,487	33,730	.073733
73	3,124,340	6.4947583	.080193	$\bar{2}$.9041365	5.3988948	250,550	2,505	31,243	.080178
74	2,873,790	6.4584550	.087040	$\bar{2}$.9397189	5.3981739	250,135	2,501	28,738	.087028
75	2,623,655	6.4189067	.094366	$\bar{2}$.9748155	5.3937222	247,584	2,476	26,237	.094371
76	2,376,071	6.3758594	.102323	$\bar{1}$.0099733	5.3858327	243,127	2,431	23,761	.102311
77	2,132,944	6.3289795	.111074	$\bar{1}$.0456124	5.3745919	236,915	2,369	21,330	.111064
78	1,896,029	6.2778450	.120817	$\bar{1}$.0821280	5.3599730	229,073	2,291	18,961	.120827
79	1,666,956	6.2219242	.131752	$\bar{1}$.1197572	5.3416814	219,625	2,196	16,670	.131734
80	1,447,331	6.1605679	.144446	$\bar{1}$.1597055	5.3202734	209,061	2,091	14,474	.144466
81	1,238,270	6.0928154	.158629	$\bar{1}$.2003826	5.2931980	196,426	1,964	12,383	.158605
82	1,041,844	6.0178027	.174338	$\bar{1}$.2413920	5.2591947	181,633	1,816	10,419	.174297
83	860,211	5.9346050	.191571	$\bar{1}$.2823298	5.2169348	164,791	1,648	8,603	.191561
84	695,420	5.8422472	.211372	$\bar{1}$.3250474	5.1672946	146,992	1,470	6,955	.211359
85	548,428	5.7391196	.235627	$\bar{1}$.3722251	5.1113447	129,224	1,292	5,485	.235552
86	419,204	5.6224254	.265675	$\bar{1}$.4243507	5.0467761	111,372	1,114	4,193	.265681
87	307,832	5.4883137	.302939	$\bar{1}$.4813552	4.9696689	93,254	933	3,079	.303020
88	214,578	5.3315852	.346861	$\bar{1}$.5401555	4.8717407	74,429	744	2,146	.346692
89	140,149	5.1465900	.395883	$\bar{1}$.5975669	4.7441569	55,483	555	1,402	.395863
90	84,666	4.9277090	.454133	$\bar{1}$.6571831	4.5848921	38,450	385	847	.454545
91	46,216	4.6647924	.533333	$\bar{1}$.7269985	4.3917909	24,649	246	462	.532466
92	21,567	4.3337897	.632911	$\bar{1}$.8013427	4.1351324	13,650	137	216	.634259
93	7,917	3.8985606	.732601	$\bar{1}$.8648675	3.7634281	5,800	58	79	.734177
94	2,117	3.3257209	.869565	$\bar{1}$.9393021	3.2650230	1,841	18	21	.857143
95	276	2.4409091	1.000000	0	2.4409091	276	3	3	1.000000

Effect on Mortality of Selection by the Insured; Mortality after the Deferred Dividend Period.

BY

ARTHUR HUNTER.

Under deferred dividend policies the insured have the right to select one of several options at the end of the deferred period. Under whole life policies the principal options usually consist of:

1. Taking the dividend in cash and continuing the policy in force;

2. Taking paid-up insurance purchased by the total cash value, including the dividend;

3. Surrendering the policy for the total cash value, including the dividend.

There are other options, such as converting the dividend into a paid-up addition or into an annuity, but these are seldom selected by the insured.

It has generally been assumed that the policyholders will exercise their option in a way which will have an effect on the mortality of the company for better or for worse, but no statistics have been published on this subject. We have recently made an investigation of the mortality under two of the options which may be exercised by the insured on ordinary life and limited payment life policies issued on standard lives in countries in which the domestic or Northern scale of premiums is charged, the great majority of the policies being on residents of the United States and Canada. The deferred dividend periods were 10, 15, and 20 years. Under limited payment life policies the dividend periods were equal to or longer than the premium-paying periods. The investigation covers the policies reaching the end of their deferred dividend periods in the years 1894 to 1909, the data being carried to the policy anniversaries in 1911.

The forms of settlement under which the mortality was investigated were:

1. Policy continued in force and accumulated dividends taken in cash.

2. Total cash value, including dividends, converted into paid-up insurance.

A medical examination had been required as a condition of converting the total cash value into paid-up insurance if the amount of the paid-up insurance was greater than that of the original policy, and accordingly the investigation under (2) was divided into two parts.

Policy Continued in Force and Dividends Taken in Cash.

The mortality after the deferred dividend period partly depends upon the percentage of policyholders who continue their insurance. If all the insured continued their policies, the mortality by amounts insured would be approximately equal to the American or to the Compound Progressive Table of the New York Life Insurance Company. If only 25 per cent. elected to keep their policies in force, taking their dividends in cash, there is a likelihood that the group would be poorer risks than the 75 per cent. who surrendered their policies for the total cash value, or selected paid-up insurance or annuities. The best form of settlement for the beneficiaries of the insured who were not in good health would be to continue the policy in force and take the dividends in cash. The first question, therefore, to be considered is the proportion of policies taking that mode of settlement to the total having the right to make a selection.

During the years 1894 to 1909, the percentage of ordinary life and limited payment life policies which were continued in force, the accumulated dividends being taken in cash, was about 25 per cent. of the total maturing policies on these plans. Since 1909 the largely increased number of policies reaching the end of their deferred dividend periods justified the expense of a special staff to look carefully into the cause of discontinuance and to try by letter or by personal interview to keep these policies in force. The result has been that the percentage continued in force (dividends being taken in cash) has gradually increased to over 65 per cent. of the policies having that option. The policies finishing their deferred dividend periods from 1894 to 1909 showed little variation in the percentage of the business continued under the option in question, so that these groups constitute a homogeneous body.

The expected deaths have been determined by the American Table, as the medical selection had taken place fifteen years, on the average, prior to the end of the period. The first exhibit shows the

experience under the ordinary life and under the limited payment life policies, the 10, 15, and 20 year dividend periods being combined:

POLICY CONTINUED IN FORCE AND DEFERRED DIVIDENDS TAKEN IN CASH. *Statistics for 10, 15, and 20 Year Periods Combined. Expected Deaths by American Table of Mortality.*

By Policies.				By Amounts Insured.		
Policy Year after Deferred Dividend Period.	Actual Deaths.	Expected Deaths.	Ratio of Actual to Expected Deaths.	Actual Deaths.	Expected Deaths.	Ratio of Actual to Expected Deaths.
Ordinary Life.						
1	127	78	163%	$603,800	$305,600	198%
2–3	185	149	124	730,300	558,000	131
4–5	141	113	125	492,700	422,700	117
6–10	145	152	95	583,400	594,800	98
11–17	39	30	130	122,500	114,100	107
Total	637	522	122%	$2,532,700	$1,995,200	127%
Limited Payment Life.						
1	73	45	162%	291,200	180,200	162%
2–3	84	83	101	368,100	335,800	110
4–5	56	56	100	287,900	233,600	123
6–10	57	61	93	290,600	231,300	126
11–17	18	16	113	65,700	62,200	106
Total	288	261	110%	$1,303,500	$1,043,100	125%
Ordinary Life and Limited Payment Life Combined.						
1	200	123	163%	895,000	485,800	184%
2–3	269	232	116	1,098,400	893,800	123
4–5	197	169	117	780,600	656,300	119
6–10	202	213	95	874,000	826,100	106
11–17	57	46	124	188,200	176,300	107
Total	925	783	118%	$3,836,200	$3,038,300	126%

The following deductions may be made from the foregoing table: 1st, that the mortality in the first policy year after the deferred dividend period is distinctly higher than for the succeeding policy years; 2d, that the effect of the selection against the company continues for fully five years; 3d, that the relative mortality is higher under ordinary life policies than under limited payment life; 4th, that the mortality by policies is slightly lower than by amounts insured.

The evidence is clear that the insured do select against the inter-

ests of the company as a whole, and for several years due provision has been made for the additional mortality foreseen, in calculating the dividend at the end of the deferred dividend period. The larger the proportion of those who continue their policies, however, the less will the selection be, and as the company has now succeeded in having a large majority of the policies continued in force, the monetary effect of the selection will doubtless hereafter be small.

The foregoing experience on the ordinary life and the limited payment life was divided into three groups according to the length of the deferred dividend period, showing results separately for 10, 15, and 20 year deferred dividend periods. As the data were few when thus divided, a synopsis only is given. Combining ordinary life and limited payment life policies, the ratio of actual to expected deaths by policies for the first five policy years after the deferred dividend period was 140 per cent. for 10 year periods, 125 per cent. for 15 year, and 126 per cent. for 20 year periods. The high mortality among the 10 year periods may be partly due to scantiness of data.

Total Cash Value Including Deferred Dividend Converted into Paid-up Insurance.

One of the options granted the insured was that of converting the total cash value including the dividends into paid-up insurance on which no future premiums would be required. If the amount of paid-up insurance was less than the face value of the original policy, no evidence of insurability was required; but otherwise a medical examination was necessary, and the insured had to be practically as good a risk as if he were applying for new insurance.

Paid-up Policies with Medical Examination.

The statistics in this group are too few to justify dividing them by individual policy years, by plans of insurance, or by dividend periods.

As the average age of the policyholders at the beginning of the above investigation was about fifty-three, and the average period of exposure over seven years, the mortality is distinctly favorable. Although the statistics are not large, they are sufficient to indicate that the medical examination was efficient, and, probably, that the type of people who desired the paid-up insurance were among the

better class of the insured. Those who voluntarily desire to convert their entire cash value including the dividends into paid-up insurance, are likely to be men in comfortable circumstances, and therefore, as a class, good risks.

EXPECTED DEATHS CALCULATED BY AMERICAN TABLE.

Policy Years after End of Deferred Dividend Period.	Actual Deaths.	Expected Deaths.	Ratio of Actual to Expected.
By Policies.			
1–5	41	47.1	87%
6–17	26	38.8	67
Total	67	85.9	78%
By Amounts Insured.			
1–5	$131,500	$194,400	68%
6–17	119,400	185,700	64
Total	$250,900	$380,100	66%

Paid-up Insurance without Evidence of Insurability.

The statistics in this group were three times as numerous as those in the group under which a medical examination was required, but as there did not seem to be any reason for anticipating a different mortality by deferred dividend period, no sub-division was made. All the original policies were on the ordinary life plan.

EXPECTED DEATHS CALCULATED BY AMERICAN TABLE.

Policy Years after End of Deferred Dividend Period.	Actual Deaths.	Expected Deaths.	Ratio of Actual to Expected Deaths.
By Policies.			
1	44	44	100%
2–5	143	151	95
6–17	88	98	90
Total	275	293	94%
By Amounts Insured.			
1	$205,600	$202,500	102%
2–5	749,200	714,800	105
6–17	361,200	436,400	83
Total	$1,316,000	$1,353,700	97%

The mortality is somewhat heavier in the early policy years than would have been expected, as the amount of the insurance was less than under the original policy.

Conclusion.

In order to see the varying effect of selection, a synopsis has been prepared for the first five policy years following the deferred dividend period, as these would be the years which would show the principal effect of selection.

Synopsis of Mortality for First Five Policy Years after Deferred Dividend Period.

Expected Deaths Calculated by American Table.

	By Policies.	By Amounts Insured.
Dividends taken in cash and policies continued..	127%	136%
Total cash value converted into paid-up insurance *without* evidence of insurability.....	96%	104%
Total cash value converted into paid-up insurance, —*with* medical examination	87%	68%

The foregoing table indicates that the policyholders make a conscious selection in determining the form of settlement best suited to their condition, and that the resulting mortality depends upon which form of settlement is most to the advantage of the insured or the beneficiary.

Provision for Expenses in Savings Contracts.

BY

MILES MENANDER DAWSON.

I.

"Le contrat naît. De suite, il va être imputé, sur le montant des premières cotisations, le prélèvement forfaitaire pour frais de gestion. Ce prélèvement atteint, en général dans les sociétés d'épargne, 10% de la souscription.

"Pour un contrat d'épargne de dix ans, la première cotisation sera donc retenue pour frais de gestion et, pour un contrat de vingt ans, les deux premières cotisations.

. . . . . .

"Qui pense un seul instant à critiquer, dans les sociétés d'assurances sur la vie, la manière dont elles calculent la valeur de rachat accordée à leurs assurés, lorsque le contrat est résilié en cours d'existence? Cette valeur chez elles est nulle jusqu'à l'expiration des deux premières années, pour augmenter petit à petit, au fur et à mesure que le contrat s'achemine vers le terme qui lui a été assigné. Et ainsi les retenues opérées sur les remboursements sont, à la troisième année, égales à deux années de primes, pour diminuer en proportion de la plus grande durée d'exécution du contrat."—From a Pamphlet by M. Georges Hamon, Professeur d'Assurances a l'Institut Commercial, Paris.*

* The following is a translation of Professor Hamon's statement in English:

"The contract comes into existence. Thereupon there must be calculated how much out of the amount of the first payments is to be applied to the payment of management expenses. This provision generally extends in savings societies to 10% of the entire amount to be paid upon the contract. For a savings contract, paid for in ten years, the first year's payment would be taken for management expenses and for a contract paid for in twenty years, the first two years' payments.

. . . . . .

"Who would for an instant think of criticizing, in life insurance companies, the manner in which they calculated the surrender values granted the policyholders in case the contract is prematurely terminated? They allow nothing until after the first two years, increasing gradually as the contract approaches maturity. In this way the surrender charge retained is, at the end of the third year, equal to two years' premiums, diminishing in proportion to shorter or longer duration of the contract."

The pamphlet from which this is taken was recently issued as a protest against proposed legislation by the French parliament prohibiting societies which solicit the investment of savings, making these large deductions from the principal sums so "invested" with them, to cover initial and general expenses.

Presumably no member of this Society would countenance such a proceeding, which is, therefore, merely put forward here as a horrible example of the consequences of ignoring ethical considerations in actuarial matters.

It is at once obvious that "investment" in a financial institution calls, first, for the integrity of the principal paid in, and that no plan which impairs the principal when paid in, by deducting a provision for expenses, is sound or admissible. In this regard a savings society or any other concern accepting such investments stands on no different footing from that of a savings bank or trust company as regards its depositors.

The actuary's skill may be utilized to confuse and bedevil this manifestly correct principle, as the skill of an attorney may be directed to defeat the administration of justice instead of to promote it, or the skill of a physician to destroy life instead of to preserve it.

The actuary's skill is brought into it in this way: On the basis of earning a certain rate of interest, "maturity" for the aggregate amount of all instalments, no interest being drawn in the meantime, will be reached at the end of a given period, (*a*) by putting aside each year an amount considerably less than the year's payments on the contract, or (*b*) by putting aside after the first year or so all the payments subsequently made on the contract but nothing during this preliminary period. It is shown by the following mathematical processes:

Let $P' =$ the actual annual payment,
$n =$ the term of years,
\$1.00 $=$ the maturity value $= nP'$,
$i =$ the rate of interest per annum.

Then

$\frac{v^n}{\mathrm{a}_{\overline{n}|}} =$ the net annual payment $= \pi$, say.

$P' - \pi =$ the annual margin, treated as set free as a provision for management expenses, if deducted in equal amounts each year.

$(P' - \pi)\mathrm{a}_{\overline{n}|}$ = present value of such maximum provision

$\dfrac{(P' - \pi)\mathrm{a}_{\overline{n}|}}{P'\mathrm{a}_{\overline{m}|}} = 1$, solved for m, will give the term during which, if all payments thereafter are applied without deduction to mature the contract, payments may be applied in toto as a provision for management expenses.

Obviously P', if paid each year and actually invested at i interest, annually compounded, would be worth $P'(s_{\overline{n+1}|} - 1)$ at the end of n years, instead of nP'. What has been taken from the "investor," therefore, for management expenses amounts to robbing him of interest on his entire principal payments at i per cent. per annum, in case he makes payments for the entire period of n years. This also is true, whether the deduction is made in equal amounts each year or by appropriating the entire payments for one or more years.

The interest, so appropriated, is usually at a rate of from 3% to 4%, sometimes even higher.

If so much of the interest actually received by the concern upon its investments, i. e., 4%, for instance, out of the 5% or even 6% realized, were so appropriated *as realized,* it is manifest that men would not with full knowledge "invest" their savings, unless lied to about "gains from lapses," etc.—in other words, induced by fraudulent misrepresentations.

Yet more assured is it that they could not with full knowledge be induced to "invest," if they understood that, as in the case cited by M. Hamon, not only would the interest to the extent of 3% or 4% on the principal per annum be appropriated for expenses if the investment were carried to maturity, but the whole amount would be deducted from the first payments made and in event at any time no further payments were forthcoming, such entire amount would be deducted in determining the proceeds of the "investment," even to the full extent of reducing such proceeds to nil.

A priori, then, such an alleged investment could be thrust upon a man of sound mind only by unconscionable fraud and by no other means whatever.

The actuary might conceivably minister to such a concern by aiding it to compute how much could thus be deducted and enough be left "to enable the company to carry out its guarantees," i. e., to repay the money "invested" with it, without interest, at

maturity. Indeed, some person must in every case, of course, perform this initial service for it. But the principal function which actuaries are sometimes called upon to perform is to value the liabilities under these so-called "investment contracts." This they are expected to do by valuing "our guarantees," i. e., computing the present value of the promise to return "at maturity" the payments actually made, with such profits, if any, as may appear, and deducting the present value of the net payments remaining to be made by the "investor."

It is an innocent-looking financial computation, susceptible of being mathematically accurate; but there the question arises: What is the actuary's responsibility for the implications?

Unquestionably, the implication of the actuary's "certificate of solvency" to the average person is that the concern, and its methods, rates and promises are sound; and since he has gone in, or is going in, on the theory that his principal is secure and that he will receive returns equal to or exceeding the usual rate of interest, he cannot suppose that a solemn farce is being perpetrated, signifying only that the concern will perhaps some day have in hand the principal he has paid, though it is now impaired. That the concern is really sound and worthy of his confidence, is the interpretation he puts on the actuary's certificate; and surely it will be well for our profession if this trust shall always be justified, so far at least as that no concern operating on an unsound financial plan shall boast such a certificate.

To be sure, upon any savings plan, the necessary expenses of investing funds and caring for investments must be a charge upon the interest earnings. Such a necessary charge, including investment losses, ranges from one eighth of one per cent to one fourth in well managed companies. This does not include anything for procurement expenses, it being rightly regarded by genuine savings institutions as vicious in the extreme either to impair the principal paid in, or even to diminish the interest, in order to remunerate agents for inducing men to "save" their money in such a way.

It is, however, not so much what actuaries might do—and have done at times—to bolster up these quasi-savings concerns, as it is the bearing of these considerations upon practices in life insurance, that prompts this paper. In the second quotation from M. Hamon's pamphlet, the argument is presented that in fixing their surrender values, life insurance companies behave in the same manner—a most serious accusation if warranted.

M. Hamon adverts only to surrender charges, but his assertions would have even greater application to policy values; accordingly, these will be considered together. Indeed, they are but two aspects of the same thing. M. Hamon's language implies that a surrender charge is of the nature of a penalty for "breaking the contract"—a view once urged but really utterly untenable since a life insurance contract is "unilateral," i. e., is an obligation upon but one party, only, viz: the insurance company. The surrender charge is fixed upon other grounds altogether, and in part, admittedly, upon the company's being unwilling that the transaction to date should inflict a loss upon it, taking into account all the costs whether mortuary or expense; precisely, for instance, as the pseudo-savings bank which has expended the first two years' payments on a twenty-year "investment" must hold out a large sum to avoid a "loss" on the transaction.

The difference lies in the justification of the expenditure, in that and nothing else.

It has already been shown that, in the case of a savings institution, such expenditure is not justifiable and, indeed, is irreconcilable with the very notion of saving money.

In life insurance, it is the protection of the insurance which is sold; investment is incidental as regards life and limited payment policies, and only collateral as regards endowment insurance policies. Life insurance is worth, and must be paid for at, its cost, including both mortuary and expense payments. The surrender values and also, in the aggregate at least, the reserves, must be so computed as to cause the withdrawing policyholders, as well as the continuing, to pay the full cost; else one class or the other would be taxed an undue proportion of the total cost.

Therefore, the man who purchases a whole life insurance policy has no more similarity to the man who deposits his money in a savings institution than, for instance, has a man who has expended his money for commodities and consumed or partly consumed the same. Whether anything can be allowed him as surrender value depends among other things upon the cost to the company of the insurance already supplied, provided such costs are proper and reasonable.

In the nature of things any genuine savings plan must attract deposits or investment by safeguarding the principal and returning a remunerative rate of interest, and is neither genuinely a savings

plan nor one which would at all attract deposits or investment upon its merits, if the principal intrusted to it, or even a big proportion of its earnings, were expended for expenses of management and procuration. On the other hand, life insurance, being an equivalent for premiums and not a mere accumulation of them, represents outlay to pay claims, to remunerate for services often difficult and expert and to recoup the number of members lost by lapse and death, in order that distribution of costs over a large number, which is essential to all insurance, may be established and maintained.

On this account, in valuing life insurance policies, the amount and incidence of disbursements, whether in payment of claims or of expenses, may properly be taken into account; and if this has not been done in computing reserves, it may with equal propriety in determining surrender charges; while in valuing the true obligations of a concern with which its customers consider that they have deposited or invested savings, precisely the contrary position must be taken.

But while the distinction is broad and deep between life insurance and savings in this regard, it is not so deep between single premium and limited premium policies or endowment insurance policies, on the one hand, and savings on the other.

The whole life policy with level premiums during life may, I think, be taken as the norm in life insurance. One year term or other short term insurance can be advocated as such; but the actual choices the world over show life insurance with the right to renew indefinitely, without increase of rate, the recognized norm. It follows that necessary initial and general expenses incurred in respect of such policies are as much a part of the cost of carrying the insurance as the payment of claims; both must be met if the insurance is to be supplied. Yet if the company is to make each bear his full share, initial expenses, general expense and mortuary cost, altogether, must not, even the first year, exceed the total premium; else they, who pay no more premiums, will escape without paying their share.

This indicates what is known as "straight modified preliminary term" valuation, advocated by Dr. Zillmer of Germany, Dr. Sprague of Great Britain and Mr. McClintock, and first actually introduced in practice by myself in 1897, as the minimum standard

compatible with entirely justifiable conditions. This view is enforced when attention is given to the other forms of life insurance.

It is not necessary to do more in the case of single premium insurance than glance at it. A mere glance shows that current practices are indefensible and that principles of professional ethics should be applied to the problem. The most usual course in fixing surrender values is to strip off the loading wholly, treating it as at once earned. But, clearly, the member has received no greater value in protection in one case than in the other. The maximum surrender charge then should not be greater than upon the whole life level premium policy in force the same number of years. The remainder should be treated purely as accumulated savings until required, and the formulas, to square with these principles, should run like this:

$$A'_x = P'_x \mathrm{a}_x,$$
$$_nVA'_x = {}_nV'_x + P'_x \mathrm{a}_{x+n},$$

in which, since P'_x is loaded, provision is made for future expenses at the same rate as on the whole life level premium plan. That is, to put it technically, "the entire loading is reserved," i. e., for the purposes of the future.

This conforms with the best practice in Great Britain and in several other countries; but in the United States it has not yet made its appearance save in a few sporadic instances.

The reasoning of it is simple. The whole life annual premium at that age is the full and sufficient rate for life insurance; the remainder of the single premium is a deposit to pay for insurance to be furnished after the first year and should be reserved for the purpose.

The same reasoning applies with even greater force to a limited premium policy. Of the full premium for the first year and every year, all above an amount equal to the whole life rate is obviously deposited for the purpose of accumulating a fund sufficient to prepay all premiums which would otherwise fall due after the expiration of the payment period. The insurance would cost not more than upon the whole life plan, and the formulas should be:

$$_mP'_x = \frac{P'_x \mathrm{a}_x}{\mathrm{a}_{x\overline{m}|}},$$
$$_nV_mP'_x = {}_nV'_x + ({}_mP'_x - P'_x)_n u_x.$$

The case is different but even plainer, if possible, as regards endowment insurance, for here life insurance adds a pure savings feature. It is obvious that the entire premium over the whole life premium at the same age is purely for savings purposes, paid to cause the policy to mature as an endowment at the end of the period; and the proper formulas become:

$$P'_{x\,\overline{m}|} = \frac{P'_x \mathrm{a}_{x\,\overline{m}|} + A_{x\,\overline{m}|}^{\,1}(1 - {}_mV'_x)}{\mathrm{a}_{x\,\overline{m}|}},$$

$${}_nVP'_{x\,\overline{m}|} = {}_nV'_x + (P'_{x\,\overline{m}|} - P'_x)\,{}_n\ddot{a}_x.$$

In the foregoing ${}_nV'_x$ has been used instead of ${}_nV_x$, to indicate that the true value of a whole life level premium policy, taking into account all costs, should be used instead of the net level premium reserve.

By the foregoing it is not meant to imply that no loading may be imposed on the purely savings portion of the premium; but if such be done, it is contrary to principle that it be used for expenses.

There have been some queer consequences in the United States at times of ignoring these equitable considerations, which will, perhaps, scarcely need more than to be mentioned, to be condemned.

Thus, for instance, some years ago a leading company actually put out 10-year endowment insurance rates that for several ages were larger than the whole life rate plus the annual rate for a simple endowment of the difference between the face of the policy and the tenth reserve on the life policy. That is:

$$P'_{x\,\overline{10}|} > P'_x + \frac{1 - {}_{10}V_x}{s_{\overline{11}|} - 1}$$

$$> P'_x + \frac{(1 - {}_{10}V_x)(1+i)^{-10}}{\mathrm{a}_{\overline{10}|}}.$$

Indeed at a few ages, in this case:

$$P'_{x\,\overline{10}|} > P'_{x\,\overline{1}|} + \frac{1}{s_{\overline{11}|} - 1}$$

$$> P'_x + \frac{v^{10}}{\mathrm{a}_{\overline{10}|}}.$$

Want of acquaintance with the ethical considerations here advanced must excuse actuaries generally, and particularly depart-

ment actuaries, for failing immediately and strongly to condemn the "endowment fund" concerns of a few years ago. They sold ten-year endowment policies, one with $500 insurance for five years and $1,000 for the second five years, and the others with return of premiums at death with 20% added. The bait was an estimate of enormous surplus earnings; the premium $100 per annum; the premium and reserve formulas in the more common case as follows:

$$P'_{x\,\overline{10}|} = \$100,$$

$$\text{Net prem. 1st year} = P^{1}_{x\,\overline{1}|}(\$120),$$

$$\text{Net renewal prem.} = \frac{120\{A_{\overline{x+1}\,|\,\overline{9}|} + (IA)_{\overline{x+1}\,|\,\overline{9}|}\} + 1000A_{x+1\,\overline{9}|}}{\mathbf{a}_{x+1\,\overline{9}|}}.$$

In other words, $100 premium the first year for $120 term insurance for one year and the accumulations of savings correspondingly robbed! Of course, the effect of this was to destroy the prospect of the policy yielding as large returns as a ten-year endowment insurance, whereas the representation was that, owing to the little or no insurance involved, the returns would be much larger.

The full preliminary term method of valuation is bad enough, in all conscience, in these regards; the following comparison shows how it discriminates in the distribution of the burden of expense costs against plans involving savings:

MARGINS SET FREE FOR FIRST YEAR EXPENSES.

(Premiums, Same as Formerly Used by N. Y. Cos.)
American Exp. Table—3½%. *Age* 35.

Whole Life.		20 Pay't. Life.		10 Pay't. Life.		20 Qear End't.		10 Year End't.	
M.P.T. & F.P.T.		M.P.T.	F.P.T.	M.P.T.	F.P.T.	M.P.T.	F.P.T.	M.P.T.	F.P.T.
$19.47		$21.96	$29.70	$21.21	$52.89	$23.37	$43.84	$31.14	$99.06

The amounts under the straight modified preliminary term (M.P.T.) method, by which the margins on limited payment and endowment premiums exceed the margin on the whole life premium for a policy at the same age, represent a higher loading upon each premium including that for the first year. This itself is unjustifiable, if it is used for expenses, as being tantamount to deducting sums each year from the portion of the premium which is purely for savings; but as regards policyholders who surrender, it pales into relative insignificance beside the outrage involved in

full preliminary term valuation. Such gross discrimination has in several states been greatly reduced by the adoption of one or another so-called "modified preliminary term" method of valuation, empirical when not based upon the solid ethical considerations here advanced; but evidently much remains to be done before valuation squares with correct principles.

Among the recent abuses of a flagrant character which have come to my attention are these:

A so-called "coupon" or premium reduction policy, paid up in 20 years. The last premium upon this was barely higher than ${}_{19}P_{x+1}$ by the standards employed. Yet valuation was asked on the full preliminary term basis and the assumption that the net premium after the first year was level at ${}_{19}P_{x+1}$. This would have thrown the loading heavily into the earlier years.

Another "coupon" or premium reduction policy called "20 payment life," in order to secure valuation as full preliminary term under the standard in a certain state. The first year's premium is equal to a 20-year endowment premium; and it may be renewed as a 20 payment life at a greatly reduced premium, or as a 20-year endowment at the same premium as the first year.

All these things indicate that it would be well worth while for actuaries to give close and careful thought to the ethical question involved. Many, indeed most, of these departures from good practices have been due to want of a clear appreciation of the fundamental principles applicable in such cases.

If the principle which has been set forth in this paper stands the test of criticism and impartial judgment, as I believe it will, the faithful application of it might have most important and, I think, widely beneficial consequences, as, for instance:

1. The payment of commissions, first year and renewal, the same *in amount* for the same amount of insurance at a given age upon the life, limited payment, or endowment plan, this amount not being larger than the whole life premium warrants.

In most European countries, and in Australia and New Zealand, the *same amount* is paid, regardless even of age, which seems unnecessary. In all countries, well-managed companies vary the rates of first-year commissions by plans so as to yield approximately the same amounts.

2. The determination of surrender values on some reasonable basis, having due regard both for preliminary and other expenses

and also for the duty of providing against the financial consequences of adverse selection.

Our laws upon this subject provide for the same surrender charge, no matter what method of reserves has been employed, which is supremely ridiculous.

3. Loadings in excess of the limited amounts, which would thus be treated as available for expenses, would be treated as for contingencies and for dividends. It is probable that something much more rational, dependable and desirable than the dividend methods now in use would result.

4. Upon limited payment policies special reserves to cover expenses (and probably also to cover contingencies and provide dividends) would pretty certainly be accumulated. This would be as in Great Britain and several other countries and would work well.

One result would be that dividends, after the payment period is completed, would be more acceptable to policyholders.

5. Endowment insurance policies, largely purchased as investments, would yield better returns because charged with as little for expense as life policies issued at the same age.

It appears, therefore, that the acceptance of the principle that no inroads can properly be made for the payment of expenses upon the principal of savings, deposited or invested, and its application to all payments to a life insurance company in excess of its life premium, is ethically sound, whether judged upon grounds of pure reason or by its actual or probable results in practice.

Total Disability Benefits: with Special Reference to the Use of Pension Fund Methods of Calculation.

BY

JAMES F. LITTLE.

Introduction.

In view of the numerous papers on the subject of disability benefits that have been presented to the Society in recent years, a further addition to the literature of the subject requires, if not an apology, at least an explanation. To meet this requirement it may be pointed out that the subject is by no means exhausted, and that some questions have been raised, the answers to which are in dispute. My principal reason, however, for venturing into the field is to present the matter from the pension fund aspect, the lives becoming disabled corresponding to retirants on pension. I think the subject is easier to understand when presented in this manner, while the resulting arithmetical operations are simpler and, perhaps, fewer than those called for by the methods now in use.

A considerable addition to the normal complexity of the subject arises from the introduction of the requirement that the American Experience mortality shall be assumed to represent the aggregate mortality of insured lives, active and disabled, during all policy years. A number of points arising out of this assumption will be dealt with in Part III—General Considerations.

In the following investigations it is assumed that a table is available showing at all ages the numbers living as active lives and the numbers becoming disabled out of a given number at the youngest age, when all are assumed to be active lives. It will also be assumed that all deaths during each year among lives active at the beginning of the year are deaths among active lives. The tabular number becoming disabled during the year will, therefore, include only those who become disabled but are alive at the end of the year. In symbols, the "active" deaths of the table are really d_x^a and not d_x^{aa} while the rate of disablement is not r_x but $p_x^{a\mathfrak{i}}$. The symbol r_x

has, however, been used in the formulas given below as a matter of convenience. This has the sanction of precedent, as in Mr. Hunter's table in *T. A. S. A.*, XII, 66, the columns headed q_x^{aa} and r_x are, strictly, columns of q_x^a and $p_x'^{ai}$. A corollary is that the disability benefits run from the end of the year of disablement. This is accurate in the case where the benefit is the waiver of a yearly premium, and in the case of other benefits, such as payment of the sum insured by instalments, the error is analogous to that involved in assuming death claims paid at the end of the year of death.

A mortality table for invalid lives will also be required, and whole life annuity-values and the ordinary commutation columns will be assumed available for both active and invalid lives.

Part I. Premiums.

In all the waiver of premium calculations in this paper, the annual premium waived is taken as 1, and, therefore, P_x, ${}_nP_x$ and $P_{x\overline{n}|}$ do not appear in the formulas, which are applicable whether the premium to be used in the calculations is the gross or net premium or some modification of either. While it may be satisfactory to assume that the net American 3% or 3½% premium represents the loss to the company upon disablement, a similar assumption would not be permissible where a net premium derived from a table involving approximately true rates of mortality and interest was the basis of the life premium calculations.

I. The single premium to provide 1 payable on disablement is

$$\begin{aligned} {}^iA_x &= \Sigma v^{n+1} \cdot {}_np_x^{aa} \cdot r_{x+n} \\ &= \frac{1}{D_x^{aa}} \Sigma v D_{x+n}^{aa} \cdot r_{x+n} \\ &= \frac{{}^rM_x^{ai}}{D_x^{aa}}, \end{aligned} \tag{1}$$

where $$v D_x^{aa} \cdot r_x = {}^rC_x^{ai},$$

and $$ {}^rC_x^{ai} + {}^rC_{x+1}^{ai} + \cdots = {}^rM_x^{ai}.$$

I^A. If the benefit of Section I is payable only upon disablement occurring before age y, the single premium becomes

$$ {}^iA_{x:\overline{y-x}|} = \frac{{}^rM_x^{ai} - {}^rM_y^{ai}}{D_x^{aa}}. \tag{2}$$

II. The single premium for an annuity due of 1 per annum for life, commencing upon disablement, is

$$
\begin{aligned}
{}^{W}A_x &= \frac{1}{D_x^{aa}} \Sigma\, {}^{r}C_{x+n}^{ai} \cdot \mathrm{a}_{x+n+1}^{i} \\
&= \frac{{}^{ra}M_x^{ai}}{D_x^{aa}}, \qquad (3)
\end{aligned}
$$

where $${}^{r}C_x^{ai} \cdot \mathrm{a}_{x+1}^{i} = {}^{ra}C_x^{ai},$$

and $${}^{ra}C_x^{ai} + {}^{ra}C_{x+1}^{ai} = {}^{ra}M_x^{ai}.$$

The proper symbol for the benefit just found is a_x^{ai}—the value of an annuity during the disabled lifetime of an active life aged x. The symbol given above is in accordance with the plan of representing all net single premiums by the general expression "A," with the necessary additional symbols for the particular benefit in question.

II^{A}. If the benefit of Section II becomes payable only upon disablement prior to age y, the single premium is

$${}^{W}A_{x:\overline{y-x}|} = \frac{{}^{ra}M_x^{ai} - {}^{ra}M_y^{ai}}{D_x^{aa}}. \qquad (4)$$

III. The annual premium payable during active life, to provide 1 upon disablement, is (see Section I)

$${}^{i}P_x = {}^{i}A_x \div \mathrm{a}_x^{aa} = \frac{{}^{r}M_x^{ai}}{N_x^{aa}}. \qquad (5)$$

III^{A}. If benefit and premium alike cease upon attainment of age y as an active life, the annual premium of the preceding section becomes

$${}_{y-x}{}^{i}P_{x:\overline{y-x}|} = \frac{{}^{r}M_x^{ai} - {}^{r}M_y^{ai}}{N_x^{aa} - N_y^{aa}}. \qquad (6)$$

IV. The annual premium, payable during active life, to provide an annuity-due of 1 per annum during disablement, is (see Section II)

$${}^{W}P_x = {}^{W}A_x \div \mathrm{a}_x^{aa} = \frac{{}^{ra}M_x^{ai}}{N_x^{aa}}. \qquad (7)$$

IV^{A}. If benefit and premium both cease upon attainment of age

y as an active life, the annual premium becomes

$$ {}^{W}P_{y-x\,x:\overline{y-x}|} = \frac{{}^{ra}M_x^{ai} - {}^{ra}M_y^{ai}}{N_x^{aa} - N_y^{aa}}. \tag{8} $$

IV^B. If benefit ceases upon attainment of age y as an active life, but premium continues during active life, the annual premium becomes

$$ {}^{W}P_{x:\overline{y-x}|} = \frac{{}^{ra}M_x^{ai} - {}^{ra}M_y^{ai}}{N_x^{aa}}. \tag{9} $$

This last formula will be required where the regular waiver of premium benefit ceases at age y, but a modified form of disability benefit continues throughout the premium-paying term. In such case the premium for the total disability benefits may, as a matter of convenience, be made payable as long as the regular premium continues. The premium for the modified disability benefit, after age y, will generally be calculated separately, as in Section VIII.

Formulas (7), (8) and (9) are applicable to waiver of premium benefits under ordinary life policies. In the case of limited payment and endowment policies, the benefit upon disablement takes the form of a temporary annuity for the unexpired portion of the original premium term.

V. The single premium for a temporary disability annuity-due, no payment being made after age $x+t-1$ in any event (t being original premium term), is

$$ \begin{aligned} {}^{W\overline{t}|}A_x &= \frac{1}{D_x^{aa}} \sum_{n=0}^{n=t-1} {}^{r}C_{x+n}^{ai} \cdot \mathrm{a}_{x+n+1:\overline{t-n-1}|}^{i} \\ &= \frac{1}{D_x^{aa}} \sum_{n=0}^{n} {}^{r}C_{x+n}^{ai} \cdot \frac{N_{x+n+1}^{i} - N_{x+t}^{i}}{D_{x+n+1}^{i}} \\ &= \frac{{}^{ra}M_x^{ai} - {}^{ra}M_{x+t}^{ai} - N_{x+t}^{i}({}^{rs}M_x - {}^{rs}M_{x+t})}{D_x^{aa}}, \end{aligned} \tag{10} $$

where

$$ \frac{{}^{r}C_x^{ai}}{D_{x+1}^{i}} = {}^{rs}C_x, $$

and

$$ {}^{rs}C_x + {}^{rs}C_{x+1} + \cdots = {}^{rs}M_x, $$

so that

$$ {}^{rs}M_x - {}^{rs}M_{x+t} = {}^{rs}C_x + {}^{rs}C_{x+1} + \cdots + {}^{rs}C_{x+t-1}. $$

V^A. Where a limiting age, y, is set for the above benefit, and $x+t \leqslant y$, the limit is inoperative. In the other case, where $x+t > y$, the single premium becomes

$$ {}^{W\overline{t}|}A_{x:\overline{y-x}|} = \frac{{}^{ra}M_x^{ai} - {}^{ra}M_y^{ai} - N_{x+t}^{i}({}^{rs}M_x - {}^{rs}M_y)}{D_x^{aa}}. \quad (11) $$

VI. The annual premium for the benefit of Section V, payable during active life within t years, is

$$ {}^{W\overline{t}|}P_x = \frac{{}^{ra}M_x^{ai} - {}^{ra}M_{x+t}^{ai} - N_{x+t}^{i}({}^{rs}M_x - {}^{rs}M_{x+t})}{N_x^{aa} - N_{x+t}^{aa}}. \quad (12) $$

VI^A. If premium and benefit both cease on attaining age y as an active life, the annual premium becomes

$$ {}^{W\overline{t}|}_{y-x}P_{x:\overline{y-x}|} = \frac{{}^{ra}M_x^{ai} - {}^{ra}M_y^{ai} - N_{x+t}^{i}({}^{rs}M_x - {}^{rs}M_y)}{N_x^{aa} - N_y^{aa}}. \quad (13) $$

VI^B. If the benefit ceases as in the preceding section, but the premium continues during active life within t years, the annual premium becomes

$$ {}^{W\overline{t}|}_{t}P_{x:\overline{y-x}|} = \frac{{}^{ra}M_x^{ai} - {}^{ra}M_y^{ai} - N_{x+t}^{i}({}^{rs}M_x - {}^{rs}M_y)}{N_x^{aa} - N_{x+t}^{aa}}. \quad (14) $$

The foregoing formulas are sufficient to enable premiums for all ordinary waiver of premium benefits to be calculated. Another class of benefit must now be considered, taking the form of a modified waiver of premium which provides for deduction from claim of the amount of premiums waived, without interest. This benefit is usually deferred, commencing at the age where the ordinary waiver of premium benefit ceases. The actual benefit upon disability consists of interest upon the premiums from date of disability to date of death or maturity.

VII. The single premium to provide interest to date of death on all the payments of an annuity-due commencing on disablement consists of the value of

d payable immediately on disablement,
$2d$ payable one year after disablement,
$3d$ payable two years after disablement,

and so on. The value is thus that of an increasing annuity-due, of first payment d, commencing on disablement, i e.,

$$\begin{aligned} {}^{Wa}A_x &= \frac{d}{D_x^{aa}} \Sigma\, {}^{r}C_{x+n}^{ai} \cdot (Ia)_{x+n+1}^{i} \\ &= \frac{{}^{rIa}M_x^{ai}}{D_x^{aa}} \cdot d, \end{aligned} \tag{15}$$

where

$${}^{r}C_x^{ai} \cdot (Ia)_{x+1}^{i} = {}^{rIa}C_x^{ai}$$

and

$${}^{rIa}C_x^{ai} + {}^{rIa}C_{x+1}^{ai} + \cdots = {}^{rIa}M_x^{ai}.$$

VII^{A}. The above benefit is usually deferred. If it comes into operation only upon disablement occurring after age y, the single premium is

$${}^{Wd}A_{\overline{y-x}|\,x} = \frac{{}^{rIa}M_y^{ai}}{D_x^{aa}} \cdot d. \tag{16}$$

VIII. The annual premium, payable during the whole of active life, for the deferred benefit of Section VII^{A}, is

$${}^{Wd}P_{\overline{y-x}|\,x} = \frac{{}^{rIa}M_y^{ai}}{N_x^{aa}} \cdot d. \tag{17}$$

In the case of limited payment policies, this form of benefit becomes a special increasing annuity-due of first payment d on disablement, increasing yearly up to age $x + t - 1$, from and after which it continues constant for life.

IX. The single premium to provide interest to date of death on all the payments of a temporary annuity-due payable during disablement up to age $x + t - 1$ inclusive, is

$$\begin{aligned} {}^{Wd}A_{x:\overline{t}|} &= \frac{d}{D_x^{aa}} \sum_{n=0}^{n=t-1} {}^{r}C_{x+n}^{ai} \cdot \frac{S_{x+n+1}^{i} - S_{x+t}^{i}}{D_{x+n+1}^{i}} \\ &= \frac{d}{D_x^{aa}} \left\{ {}^{rIa}M_x^{ai} - {}^{rIa}M_{x+t}^{ai} - S_{x+t}^{i} \sum_{n=0}^{n=t-1} \frac{{}^{r}C_{x+n}^{ai}}{D_{x+n+1}^{i}} \right\} \\ &= \frac{{}^{rIa}M_x^{ai} - {}^{rIa}M_{x+t}^{ai} - S_{x+t}^{i}({}^{rs}M_x - {}^{rs}M_{x+t})}{D_x^{aa}} \cdot d. \end{aligned} \tag{18}$$

after the manner of Section V.

IX^{A}. Where the foregoing benefit is deferred to age y, y being

necessarily less than $x + t$, the single premium becomes

$$ {}^{Wd}A_{\overline{y-x}|\,x:\overline{t}|} = \frac{{}^{rIa}M_y^{ai} - {}^{rIa}M_{x+t}^{ai} - S_{x+t}^{i}({}^{rs}M_y - {}^{rs}M^{x+t})}{D_x^{aa}} \cdot d. \quad (19) $$

X. The annual premium, payable during active life within t years, for the deferred benefit of Section IX^A, is

$$ {}^{Wd}_{t}P_{\overline{y-x}|\,x:\overline{t}|} = \frac{{}^{rIa}M_y^{ai} - {}^{rIa}M_{x+t}^{ai} - S_{x+t}^{i}({}^{rs}M_y - {}^{rs}M_{x+t})}{N_x^{aa} - N_{x+t}^{aa}} \cdot d. \quad (20) $$

In the case of endowments, the benefit is an increasing temporary annuity-due of first payment d, commencing upon disablement, and terminating with the maximum payment at age $x + t - 1$. It differs from the one considered relating to limited-payment policies, in that all payments falling due from age $x + t$ and upwards must be excluded.

XI. The single premium for a temporary increasing annuity-due of d, payable during disablement to age $x + t - 1$ inclusive, is

$$ {}^{W\overline{t}|d}A_{x:\overline{t}|} = \frac{d}{D_x^{aa}} \sum_{n=0}^{n=t-1} {}^{r}C_{x+n}^{ai} \frac{S_{x+n+1}^{i} - S_{x+t}^{i} - (t-n-1)N_{x+t}^{i}}{D_{x+n+1}^{i}} $$

$$ = \frac{{}^{rIa}M_x^{ai} - {}^{rIa}M_{x+t}^{ai} - S_{x+t}^{i}({}^{rs}M_x - {}^{rs}M_{x+t}) - N_{x+t}^{i}(t\,{}^{rs}M_x - {}^{rs}R_x + {}^{rs}R_{x+t})}{D_x^{aa}} \cdot d. \quad (21) $$

For the last term of the numerator we have

$$ \sum_{n=0}^{n=t-1} \frac{C_{x+n}^{ai}}{D_{x+n+1}^{i}} \cdot (t-n-1) \cdot N_{x+t}^{i} = N_{x+t}^{i} \sum_{n=0}^{n=t-1} (t-n-1)\,{}^{rs}C_{x+n} $$

and

$$ \sum_{n=0}^{n=t-1} (t-n-1)\,{}^{rs}C_{x+n} = \overline{t-1}\,{}^{rs}C_x + \overline{t-2}\,{}^{rs}C_{x+1} + \cdots + {}^{rs}C_{x+t-2} $$

$$ = (t-1)({}^{rs}C_x + {}^{rs}C_{x+1} + \cdots) $$

$$ - \{{}^{rs}C_{x+1} + 2\,{}^{rs}C_{x+2} + \cdots $$

$$ + (t-1)({}^{rs}C_{x+t-1} + {}^{rs}C_{x+t} + \cdots)\} $$

$$ = (t-1)\,{}^{rs}M_x - {}^{rs}R_{x+1} + {}^{rs}R_{x+t}, $$

$$ = t\,{}^{rs}M_x - {}^{rs}R_x + {}^{rs}R_{x+t}, $$

where

$$^{rs}R_x = {}^{rs}M_x + {}^{rs}M_{x+1} + \cdots = {}^{rs}C_x + 2\,{}^{rs}C_{x+1} + 3\,{}^{rs}C_{x+2} + \cdots.$$

XI^A. Where the benefit of Section XI is deferred to age y, the single premium becomes

$$W\overline{t}|^d A_{\overline{y-x}|\,x:\overline{t}|} = \frac{{}^{rIa}M_y^{ai} - {}^{rIa}M_{x+t}^{ai} - S_{x+t}^{i}({}^{rs}M_y - {}^{rs}M_{x+t}) - N_{x+t}^{i}(x+t-y\,{}^{rs}M_y - {}^{rs}R_y + {}^{rs}R_{x+t})}{D_x^{aa}} \cdot d. \quad (22)$$

XII. The annual premium, payable during active life within t years, for the deferred benefit of Section XI^A, is

$$W\overline{t}|^d {}_t P_{\overline{y-x}|\,x:\overline{t}|} = \frac{{}^{rIa}M_y^{ai} - {}^{rIa}M_{x+t}^{ai} - S_{x+t}^{i}({}^{rs}M_y - {}^{rs}M_{x+t}) - N_{x+t}^{i}(x+t-y\,{}^{rs}M_y - {}^{rs}R_y + {}^{rs}R_{x+t})}{N_x^{aa} - N_{x+t}^{aa}} \cdot d. \quad (23)$$

The last series of benefits for which I propose to present formulas is that where the sum insured is made payable in annual instalments—10 or 20—upon disablement. This benefit may take either of the following forms:

1. An annuity-due certain for k years of $1/k$th of the sum insured, usually as an alternative to the sum insured payable at death, premiums being waived in either case, or

2. Payment of the sum insured by instalments as in (1), except that the amount of outstanding instalments is payable immediately upon death or maturity date of original contract.

Mr. E. B. Fackler has dealt with the first of these forms (*T. A. S. A.*, XII, 241), but apparently on the assumption that the instalments, forming an annuity certain for 20 years, *must* be taken upon disablement, except when the surrender value is greater than the value of the instalments. The resulting premiums for the entire contract are little in excess of the premiums for regular life policies, and the excess is less than the extra called for by the ordinary waiver of premium benefit.

Where the insured has the option of selecting paid-up insurance (for the full amount) or the annuity-certain, as the premium is waived in either case, it seems necessary to treat the waiver of

premium as the minimum benefit, and to endeavor to find the additional value of the option. There is little doubt that there will be a very decided selection against the company among the disabled, who presumably will not be compelled to make their choice till some time after disablement has been established. Where disablement is the result of disease, and death is practically certain within a few years, the instalments will not be selected, a policy loan being obviously a better way of raising immediate cash, if necessary. Where an accident not greatly reducing the prospects of longevity has caused the disability, the instalments will often prove attractive, especially where they extend over no more than ten years. It is, therefore, unsafe to base calculations for this benefit on the assumption that those who select it will be average invalid lives.

In the absence of data enabling an accurate estimate of the value of this option to be made, an empirical method of assessment must be resorted to. It seems reasonable to suppose that the bulk of those selecting the instalment benefit will survive at least one year from disablement. The value of an annuity on a disabled life one year after disablement is, at most ages, much greater than the value a year earlier, at the time of disablement. For lives selecting the instalment benefit it might be assumed that these values are equal; in symbols, that

$$\mathrm{a}'^{i}_{[x]+0} = \mathrm{a}^{i}_{[x]+1}$$

the accent indicating a special class of disabled at the time of disablement. In the formulas of this paper, aggregate (or ultimate) mortality tables for all lives, active and disabled, have been tacitly assumed (the question of select tables being dealt with in Part III—General Considerations), but this is not inconsistent with the use of a select mortality table for disabled lives in order merely to obtain the value of the option being discussed. Using the table of annuity-values supplied by Mr. Mead (*T. A. S. A.*, XII, 83), we have

$$\mathrm{a}^{i}_{[30]+1} = 11.446 = \mathrm{a}'^{i}_{[30]+0}$$

and

$$\mathrm{a}^{i}_{[50]+1} = 8.274 = \mathrm{a}'^{i}_{[50]+0}.$$

As these disabled lives, special at time of disablement, are assumed

to be average disabled lives a year later, so that

$$\mathbf{a}'^{i}_{[x]+1} = \mathbf{a}^{i}_{[x]+1},$$

we find that $$q'^{i}_{[30]} = .06,$$

and $$q'^{i}_{[50]} = .09,$$

rates of mortality quite as high, in my opinion, as are likely to prevail during the first year of disablement among those choosing the instalment option.

Upon whatever plan the value of such a disabled life is assessed, a table of $A'^{i}_{x]}$ must be constructed, when the value of instalment option, *as a benefit additional to the regular value of the waiver of premium* is, per unit of sum insured,

$$\frac{\mathbf{a}_{\overline{k}|}}{k} - A'^{i}_{[x]}.$$

One difficulty remains. The comparatively low rate of mortality for the first year of invalidity among those selecting the instalment option necessarily implies that these are a relatively small proportion of the total becoming disabled. Probably this proportion would never exceed one fifth, but the actual figure at which it must be placed is a matter for careful individual judgment. Though the proportion would doubtless vary with age at disablement, a uniform ratio for all ages may be used in practice. In the following formulas, B represents the assumed fraction of the total becoming disabled who select the instalment option, the value of the option per person disabled being

$$\left(\frac{\mathbf{a}_{\overline{k}|}}{k} - A'^{i}_{[x]}\right)B.$$

XIII. The single premium for the option of selecting upon disablement an annuity-certain of $1/k$th of the sum insured per annum for k years, in lieu of fully paid-up insurance, is, per unit of sum insured,

$$\begin{aligned} {}^{Ik}A_x &= \frac{1}{D^{aa}_x}\Sigma^r C^{ai}_{x+n}\left(\frac{\mathbf{a}_{\overline{k}|}}{k} - A'^{i}_{[x+n+1]}\right)B \\ &= \frac{{}^{rIk}M^{ai}_x}{D^{aa}_x}, \end{aligned} \qquad (24)$$

where $${}^{r}C_{x}^{ai}\left(\frac{\mathrm{a}_{\overline{k}|}}{k}-A'^{i}_{[x+1]}\right)B={}^{rIk}C_{x}^{ai},$$

and $${}^{rIk}C_{x}^{ai}+{}^{rIk}C_{x+1}^{ai}+\cdots={}^{rIk}M_{x}^{ai}.$$

XIII^{A}. Where this benefit ceases upon attainment of age y as an active life, the single premium becomes

$${}^{Ik}A_{x:\overline{y-x}|}=\frac{{}^{rIk}M_{x}^{ai}-{}^{rIk}M_{y}^{ai}}{D_{x}^{aa}}. \tag{25}$$

XIV. The annual premium, payable during active life, for the benefit of Section XIII, is

$${}^{Ik}P_{x}=\frac{{}^{rIk}M_{x}^{ai}}{N_{x}^{aa}}. \tag{26}$$

XIV^{A}. The annual premium, payable during active life up to age y, for the benefit of Section XIII^{A}, is

$${}^{Ik}_{y-x}P_{x:\overline{y-x}|}=\frac{{}^{rIk}M_{x}^{ai}-{}^{rIk}M_{y}^{ai}}{N_{x}^{aa}-N_{y}^{aa}}. \tag{27}$$

XIV^{B}. In the case of limited payment policies, the premium may cease at age $x+t$, and the benefit continue thereafter to age y. In this case the annual premium becomes

$${}^{Ik}_{t}P_{x:\overline{y-x}|}=\frac{{}^{rIk}M_{x}^{ai}-{}^{rIk}M_{y}^{ai}}{N_{x}^{aa}-N_{x+t}^{aa}}. \tag{28}$$

In the case of endowment policies, except those for very long terms, this option will have little or no value after the first few years of insurance. Selection will not operate so strongly as in the case of life policies, and it seems scarcely worth while to present formulas when actual premiums must be fixed almost arbitrarily. Remembering that only the excess over the waiver of premium benefit is in question it would usually be safe to ignore this benefit altogether.

If the instalment benefit is of the second form, with the outstanding balance of the sum insured payable at death or maturity, the excess of the value of the disability benefits over the waiver of premium consists in interest on the instalments from date of payment to date of death or maturity. If the number of instal-

ments is k, the benefit, for a sum insured of a unit, is an annuity-due commencing on disablement at d/k and increasing by a like amount yearly up to the kth payment, after which it remains constant.

XV. The single premium for the above described benefit, under an ordinary life or limited-payment policy, is

$$\begin{aligned} {}^{Ek}A_x &= \frac{d}{kD_x^{aa}} \Sigma\, {}^{r}C_{x+n}^{ai} \frac{\mathbb{S}_{x+n+1}^{i} - \mathbb{S}_{x+n+k+1}^{i}}{D_{x+n+1}^{i}} \\ &= \frac{d}{kD_x^{aa}} \Sigma\, {}^{rs}C_{x+n}(\mathbb{S}_{x+n+1}^{i} - \mathbb{S}_{x+n+k+1}^{i}) \\ &= \frac{{}_{k}^{rs}M_x}{D_x^{aa}} \cdot \frac{d}{k}, \end{aligned} \tag{29}$$

where

$$(\mathbb{S}_{x+n+1}^{i} - \mathbb{S}_{x+n+k+1}^{i})$$

is formed for the single value of k required, and

$${}^{rs}C_x(\mathbb{S}_{x+1}^{i} - \mathbb{S}_{x+k+1}^{i}) = {}_{k}^{rs}C_x,$$

and

$${}_{k}^{rs}C_x + {}_{k}^{rs}C_{x+1} + \cdots = {}_{k}^{rs}M_x.$$

XV^{A}. Where this benefit ceases upon attainment of age y as an active life, the single premium becomes

$${}^{Ek}A_{x:\overline{y-x}|} = \frac{{}_{k}^{rs}M_x - {}_{k}^{rs}M_y}{D_x^{aa}} \cdot \frac{d}{k}. \tag{30}$$

XVI. The annual premium payable during active life for the benefit of Section XV is

$${}^{Ek}P_x = \frac{{}_{k}^{rs}M_x}{N_x^{aa}} \cdot \frac{d}{k}. \tag{31}$$

XVI^{A}. If benefit and premium both cease upon survival as an active life to age y, the annual premium becomes

$${}_{y-x}^{Ek}P_{x:\overline{y-x}|} = \frac{{}_{k}^{rs}M_x - {}_{k}^{rs}M_y}{N_x^{aa} - N_y^{aa}} \cdot \frac{d}{k}. \tag{32}$$

XVI^{B}. In the case of limited payment policies the benefit may cease at age y and the premium earlier, at age $x + t$, when the an-

nual premium is

$$ {}^{Ek}_{t}P_{x:\overline{y-x|}} = \frac{{}^{rs}_{k}M_x - {}^{rs}_{k}M_y}{N^{aa}_{x} - N^{aa}_{x+t}} \cdot \frac{d}{k}. \tag{33} $$

It is unnecessary to give the formula for the case where the premium is payable for life or for a term exceeding that for which the benefit is available.

XVII. In the case of endowment insurances the benefit differs from that under whole life policies in that all payments under the special annuity falling due from age $x+t$ must be excluded. So far as concerns payments in respect of disability occurring after age $x+t$, the method of Section XV[A] is applicable, and it remains only to find an expression for the value of the payments falling due from age $x+t$, in respect of disability occurring prior thereto. The value sought is, per unit of first payment of annuity,

$$ \begin{aligned} &\frac{1}{D^{aa}_{x}}\left\{\left(\sum_{n=0}^{n=t-k-1} {}^{r}C^{ai}_{x+n}\frac{kN^{i}_{x+t}}{D^{i}_{x+n+1}}\right) + N^{i}_{x+t}\left(\frac{\overline{k-1}\,{}^{r}C^{ai}_{x+t-k}}{D^{i}_{x+t-k+1}}\right.\right. \\ &\qquad\left.\left. + \frac{\overline{k-2}\,{}^{r}C^{ai}_{x+t-k+1}}{D^{i}_{x+t-k+2}} + \cdots + \frac{{}^{r}C^{ai}_{x+t-2}}{D^{i}_{x+t-1}}\right)\right\} \\ &= \frac{N^{i}_{x+t}}{D^{aa}_{x}}\left\{\left(\sum_{n=0}^{n=t-1} {}^{rs}C_{x+n}\cdot k\right)\right. \\ &\qquad\left. - ({}^{rs}C_{x+t-k} + 2\,{}^{rs}C_{x+t-k+1} + \cdots + k\,{}^{rs}C_{x+t-1})\right\} \\ &= \frac{N^{i}_{x+t}\{({}^{rs}M_x - {}^{rs}M_{x+t})k - ({}^{rs}R_{x+t-k} - {}^{rs}R_{x+t} - k\,{}^{rs}M_{x+t})\}}{D^{aa}_{x}} \\ &= \frac{N^{i}_{x+t}(k\,{}^{rs}M_x - {}^{rs}R_{x+t-k} + {}^{rs}R_{x+t})}{D^{aa}_{x}} \end{aligned} $$

and the total single premium becomes

$$ {}^{E\bar{t}|k}A_{x:\overline{t|}} = \frac{{}^{rs}_{k}M_x - {}^{rs}_{k}M_{x+t} - N^{i}_{x+t}(k^{rs}M_x - {}^{rs}R_{x+t-k} + {}^{rs}R_{x+t})}{D^{aa}_{x}} \cdot \frac{d}{k}. \tag{34} $$

Where this benefit ceases upon survival to age y as an active life, y being less than $x+t$, if $x+t-y$ is not less than k, the deferred

benefit that must be deducted is represented by

$$\frac{{}^{rs}_{k}M_y - {}^{rs}_{k}M_{x+t} - N^i_{x+t}(k\,{}^{rs}M_y - {}^{rs}R_{x+t-k} + {}^{rs}R_{x+t})}{D^{aa}_x} \cdot \frac{d}{k},$$

whence we have

$$E\overline{t}|_k A_{x:\overline{y-x}|} = \frac{{}^{rs}_{k}M_x - {}^{rs}_{k}M_y - N^i_{x+t} \cdot k({}^{rs}M_x - {}^{rs}M_y)}{D^{aa}_x} \cdot \frac{d}{k}. \quad (35)$$

If $x + t - y$ is less than k, the deduction is

$$\frac{{}^{rs}_{k}M_y - {}^{rs}_{k}M_{x+t} - N^i_{x+t}(\overline{x+t-y}\,{}^{rs}M_y - {}^{rs}R_y + {}^{rs}R_{x+t})}{D^{aa}_x} \cdot \frac{d}{k}$$

and

$$E\overline{t}|_k A_{x:\overline{y-x}|} = \frac{{}^{rs}_{k}M_x - {}^{rs}_{k}M_y - N^i_{x+t}(k\,{}^{rs}M_x - \overline{x+t-y}\,{}^{rs}M_y - {}^{rs}R_{x+t-k} + {}^{rs}R_y)}{D^{aa}_x} \cdot \frac{d}{k}. \quad (35A)$$

The corresponding annual premiums payable for any required term may be obtained in the same manner as in the numerous examples already given.

The foregoing formulas cover all the principal disability benefits now being granted. All the functions involved are of the single column order, and single and annual premiums may be calculated for all ages at entry and all periods of limitation or deferment of benefits, with a single set of tables. Some of the formulas appear rather lengthy, but they have been kept in actual working form, instead of being presented in a form apparently simpler, but not directly applicable for actual calculation without considerable addition to the extent of the auxiliary tables.

It will be convenient at this stage to recapitulate all the functions necessary for the use of the foregoing formulas. The following is a full list, and the connecting equations sufficiently indicate the methods of calculation of the various columns.

ORDINARY FUNCTIONS.

Symbol.	Equivalent.
D^{aa}_x	$= v^x \cdot l^{aa}_x$
N^{aa}_x	$= \Sigma D^{aa}_x$
D^i_x	$= v^x \cdot l^i_x$
N^i_x	$= \Sigma D^i_x$

$$S^i_x = \Sigma N^i_x$$
$$\mathrm{a}^i_x = N^i_x \div D^i_x$$
$$(I\mathrm{a})^i_x = S^i_x \div D^i_x$$

SPECIAL FUNCTIONS.

Symbol. Equivalent.

$$^{r}C^{ai}_{x} = v^{x+1} \cdot i_x = v \cdot D^{aa}_{x} \cdot r_x$$
$$^{r}M^{ai}_{x} = \Sigma\, ^{r}C^{ai}_{x}$$
$$^{r\mathrm{a}}C^{ai}_{x} = {}^{r}C^{ai}_{x} \cdot \mathrm{a}^i_{x+1}$$
$$^{r\mathrm{a}}M^{ai}_{x} = \Sigma\, ^{r\mathrm{a}}C^{ai}_{x}$$
$$^{rs}C_x = {}^{r}C^{ai}_{x} \div D^i_{x+1}$$
$$^{rs}M_x = \Sigma\, ^{rs}C_x$$
$$^{rs}R_x = \Sigma\, ^{rs}M_x$$
$$^{rI\mathrm{a}}C^{ai}_{x} = {}^{r}C^{ai}_{x} \cdot (I\mathrm{a})^i_{x+1}$$
$$^{rI\mathrm{a}}M^{ai}_{x} = \Sigma\, ^{rI\mathrm{a}}C^{ai}_{x}$$
$$^{rI^k}C^{ai}_{x} = {}^{r}C^{ai}_{x}\left(\frac{\mathrm{a}_{\overline{k}|}}{k} - A'^{\,i}_{[x+1]}\right)B$$

where k = number of instalments in which sum insured is payable on disablement,

$$^{rI^k}M^{ai}_{x} = \Sigma\, ^{rI^k}C^{ai}_{x}$$
$$^{rs}_{k}C_x = {}^{rs}C_x(S^i_{x+1} - S^i_{x+k+1})$$
$$^{rs}_{k}M_x = \Sigma\, ^{rs}_{k}C_x.$$

In all, thirteen special commutation columns are called for, seven of which are merely continuous summations of other columns, and none of which presents any difficulty in construction.

PART II—RESERVES.

Active Lives.

Two general formulas may be used for reserves. The first, which is restricted to ordinary life contracts, is

$$(P_{+n} - P)\,\mathrm{a}_{+n},$$

and is applicable whether premium and benefit both continue during active life, or either or both cease at a fixed age, the annuity in such cases being taken for the unexpired premium-term.

The alternative formula, applicable throughout, is $A_{+n} - P\mathrm{a}_{+n}$. The negative term is always readily available. Values of tempo-

rary annuities on active lives, for the various ages and terms required, must be prepared, and with these in hand the value of future premiums may be obtained in all cases.

The single premiums—values of benefits—for ordinary life policies may be obtained by the formulas supplied, and in the case of instalment benefits the same values apply to limited payment policies. In other cases the value of the benefit is a function of both duration and age, and when values for all durations are required the formulas given for the single premiums are not always the most convenient. It may happen that premiums are required when reserves are not, or are required without waiting for the more extensive operations involved in the calculation of reserves, and the formulas for single and annual premiums have been constructed so as to obtain the required values with a minimum of labor.

I shall now proceed to explain how the values of the benefits for all durations may be obtained for limited payment and endowment policies, where the benefit consists in waiver of premium either with or without deduction from claim of the premiums waived.

XVIII. For the value of the ordinary waiver of premium benefit in the case of limited payment and endowment policies, we may use the first expression given in Section V, namely,

$$\frac{1}{D_x^{aa}} \sum_{n=0}^{n=t-1} {}^{r}C_{x+n}^{ai} \cdot \mathrm{a}_{x+n+1:\overline{t-n-1}|}^{i},$$

that is

$$\frac{1}{D_x^{aa}} \left({}^{r}C_x^{ai} \cdot \mathrm{a}_{x+1:\overline{t-1}|}^{i} + {}^{r}C_{x+1}^{ai} \cdot \mathrm{a}_{x+2:\overline{t-2}|}^{i} + \cdots + {}^{r}C_{x+t-2}^{ai} \right);$$

a year later the value is

$$\frac{1}{D_{x+1}^{aa}} \left({}^{r}C_{x+1}^{ai} \cdot \mathrm{a}_{x+2:\overline{t-2}|}^{i} + \cdots + {}^{r}C_{x+t-2}^{ai} \right),$$

and so on. Hence if the values of ${}^{r}C_{x+n}^{ai} \cdot \mathrm{a}_{x+n+1:\overline{t-n-1}|}^{i}$ are formed for values of n up to the maximum value required, and summed continuously from the bottom upwards, multiplication of this latter column by the corresponding values of $(D_{x+n}^{aa})^{-1}$ gives the required series of single premiums. The values of $\mathrm{a}_{x+n:\overline{t-n}|}^{i}$ will be available, as they will be required for the valuation of benefits in possession on disabled lives.

As these values will be obtained for an annual premium to be waived of 1, the value at n years duration for term t and entry age x will be the same as for duration $n-5$, term $t-5$, and entry age $x+5$. A single table covering the maximum terms and all entry ages required will suffice for endowment and limited payment policies at all ages and durations, and for all original terms. Multiplication by the actual premium to be waived will supply the values per $1,000 of insurance.

XVIII[A]. Where the benefit ceases upon attaining age y as an active life, y being less than $x+t$, the series shown above will run from $n=0$ to $n=y-x-1$ (instead of $n=t-1$), and there will be no values for ages attained in excess of y. Otherwise the method of calculation will be identical with that of the preceding section.

It should be observed that in the case of a limited payment or endowment policy for t years, the waiver of premium benefit extends for $t-1$ years only, the value for the tth year ($n=t-1$) being zero, as all premiums will have fallen due, but when a limiting age y is introduced ($y<x+t$) the benefit is operative for $y-x$ years and not for $y-x-1$ years only.

XIX. When the waiver of premium benefit is subject to ultimate deduction of the premiums waived from the claim, for limited payment policies we have, from Section IX, the expression,

$$\frac{d}{D_x^{aa}}\sum_{n=0}^{n=t-1} {}^{r}C_{x+n}^{ai}\frac{S_{x+n+1}^{i}-S_{x+t}^{i}}{D_{x+n+1}^{i}}=\frac{d}{D_x^{aa}}\sum_{n=0}^{n=t-1} {}^{rs}C_{x+n}(S_{x+n+1}^{i}-S_{x+t}^{i}).$$

Forming first the values of $S_{x+n+1}^{i}-S_{x+t}^{i}$, by the deduction of S_{x+t}^{i} from the successive values of S_x^i, the values of ${}^{rs}C_{x+n}(S_{x+n+1}^{i}-S_{x+t}^{i})$ may be obtained for all required values of n. Substituting this function for ${}^{r}C_{x+n}^{ai}\cdot a_{x+n+1:\overline{t-n-1}|}$, the remaining processes are identical with those described in Section XVIII. The factor $\frac{d}{D_x^{aa}}$ may be used instead of $(D_x^{aa})^{-1}$, or d may be ignored and the resulting values multiplied by $d\pi$, where π is premium waived.

Where a limiting age y is introduced, exactly the same modification as was outlined in Section XVIII[A] will apply.

A single table of values will also suffice for all premium terms, the arguments $(x+t)$ and $(x+n)$ applying irrespective of the values of t and n.

XX. For endowment insurances, from the above values of $S^i_{x+n+1} - S^i_{x+t}$, must be deducted $(t-n-1)N^i_{x+t}$, prior to multiplication by ${}^{rs}C_{x+n}$. Otherwise the methods of Section XIX apply throughout.

The benefits of Sections XIX and XX usually commence at, instead of ceasing upon a limiting age, y. In such case the methods described will apply from age attained y upwards. For all earlier ages attained, the value at age y must be multiplied by $\frac{D^i_y}{D^i_{x+n}}$. During this earlier period, the values may also be found by accumulating the premium, the general formula being

$$\frac{N^{aa} - N^{aa}_{+n}}{D^{aa}_{+n}} \cdot P.$$

XXI. Instalment benefits of the annuity-certain variety require no special mention, except in the case of endowments. No formula was given for premiums in these cases, and as to reserves, it may safely be assumed that the rapid diminution and early disappearance of the value of the benefit would result in negative reserves from the outset, but of such extremely small extent as to be not worth considering.

XXII. In the case of instalment benefits of the second form, only endowment policies call for attention. Setting out the value of the benefit according to each possible year of disablement we have,

$$\begin{aligned}\frac{d}{kD^{aa}_x}\Bigg\{ {}^rC^{ai}_x \cdot \frac{S^i_{x+1} - S^i_{x+k+1} - kN^i_{x+t}}{D^i_{x+1}} \\ + {}^rC^{ai}_{x+1} \cdot \frac{S^i_{x+2} - S^i_{x+k+2} - kN^i_{x+t}}{D^i_{x+2}} \\ \cdots\cdots\cdots\cdots \\ + {}^rC^{ai}_{x+t-k-1} \cdot \frac{S^i_{x+t-k} - S^i_{x+t} - kN^i_{x+t}}{D^i_{x+t-k}} \\ + {}^rC^{ai}_{x+t-k} \cdot \frac{S^i_{x+t-k+1} - S^i_{x+t} - \overline{k-1}N^i_{x+t}}{D^i_{x+t-k+1}} \\ \cdots\cdots\cdots\cdots \\ + {}^rC^{ai}_{x+t-2} \cdot \frac{S^i_{x+t-1} - S^i_{x+t} - N^i_{x+t}}{D^i_{x+t-1}} \Bigg\}.\end{aligned}$$

The $t-1$ terms must first be calculated, the method being sufficiently obvious. Omitting the factor d/k, the remaining operations follow the plan of Section XVIII. The factor $\frac{d}{kD_x^{aa}}$ should be substituted for $(D_x^{aa})^{-1}$. The modifications already discussed apply in connection with a limiting age less than $x+t$, and a single table will cover all original terms.

Invalid Lives.

XXIII. In the case of ordinary waiver of premium benefits, the reserve at the end of any policy year is obviously the present value of an annuity-due of the amount of premium waived for the unexpired term, such annuity being calculated on the basis of invalid mortality.

XXIV. Where premiums are waived, but the amount thereof deducted from claim, from the value found as in Section XXIII must be deducted the value of an insurance of the amount of premiums waived up to date, increasing by the amount of future premiums as at the original due dates. It is unnecessary to give formulas.

XXV. When a policy carrying an instalment benefit in the form of an annuity-certain has been resolved into a definite contract for such annuity, it would at first sight appear obvious that the proper total reserve to be set against such policy is the present value of the instalments remaining to be paid. Practically, this is the reserve that must be carried, but it is not quite correct theoretically. This will be clear when it is remembered that the regular reserve is sufficient when such policy is included with the general mass, and the regular reserve is, therefore, more than sufficient when such worse than average life is withdrawn. Formulas for the amount by which the value of future instalments might be modified may readily be obtained, but the matter is practically unimportant.

XXVI. The same point arises in the case of the second form of instalment benefit. Ignoring this point, the reserve is simply the present value of a series of paid-up endowment insurances, for 0, 1, 2, ···, etc., years.

Part III—General Considerations.

This division of the subject might naturally have been expected to come first, but it seemed to me advisable to present the matter

in its simplest aspect, and to postpone for separate discussion debatable or complex features.

In connection with the formation of a combined mortality and disability table, it may be noted that if rates of mortality and invalidity are being extracted from a single experience, it is practicable so to arrange the material as to obtain p_x^{ai} and q_x^a or r_x and q_x^{aa}, affording the convenient relation

$$l_{x+1}^{aa} = l_x^{aa}(1 - p_x^{ai} - q_x^a),$$

or

$$l_{x+1}^{aa} = l_x^{aa}(1 - q_x^{aa} - r_x).$$

In general these are not available. Mr. Mead (*T. A. S. A.*, XI, 304) deals adequately with the necessary practical processes. Among his formulas is one which may be used where "absolute" rates of mortality, $q_x'^{aa}$, and invalidity, r_x' are known. The symbol $q_x'^{aa}$ represents the rate of mortality among active lives when the rate of invalidity is zero. In such a case, on the usual assumption as to the distribution of deaths, the years of life between ages x and $x+1$ of l_x^{aa} persons entering the year of age total $l_x^{aa}(1 - \frac{1}{2}q_x'^{aa})$. If this body of lives becomes subject to invalidity the expression just found is manifestly the exposed to risk of invalidity. The number becoming invalid being i_x, the absolute rate of invalidity is

$$r_x' = \frac{i_x}{l_x^{aa}(1 - \frac{1}{2}q_x'^{aa})} = \frac{r_x}{1 - \frac{1}{2}q_x'^{aa}}.$$

Similarly

$$q_x'^{aa} = \frac{q_x^{aa}}{1 - \frac{1}{2}r_x'},$$

whence

$$l_{x+1}^{aa} = l_x^{aa}(1 - r_x' - q_x'^{aa} + r_x'q_x'^{aa}). \qquad (36)$$

Mr. Mead writes

$$l_{x+1}^{aa} = l_x^{aa}\left(1 - \frac{r_x' - q_x'^{aa} + r_x'q_x'^{aa}}{1 - \frac{1}{4}r_x' \cdot q_x'^{aa}}\right),$$

and regards the change as a correction. Mr. Mead's formula, however, is merely a variant, founded on a slightly different assumption, namely, that the deaths among active lives only are evenly distributed throughout the year. Of the lives becoming disabled and subsequently dying within the year, approximately three quarters will die during the second half of the year, and Mr. Mead's

assumption is therefore not consistent with the usual one that all the deaths of those entering on the year are evenly distributed. In accordance with his assumption Mr. Mead writes,

$$r'_x = \frac{i_x}{l^{aa}_x - \frac{1}{2}d^{aa}_x}.$$

When r'_x approaches unity at the very old ages, the denominator of the right-hand expression is too small, as the number of active lives at risk is being very rapidly reduced and the actual deaths among active lives form a descending series with much more than half the total within the first half of the year. Even so, the practical error is quite unimportant.

It was stated in the Introduction that a good deal of complexity had been imported into the subject by the introduction of the requirement that the total mortality of all lives, active and disabled combined, should at all ages and durations coincide with the American Experience Table. If this table were a true select table, this requirement would be natural and proper, but as it is an ultimate table inconsistencies inevitably result. For instance, persons insuring at age 30 being all active lives and subject to the American mortality, the rate for active lives aged 30 in the first policy year is equal to the full American q_{30}. The survivors from entrants at age 29, however, will include some disabled lives subject to a high rate of mortality, and the active lives must consequently be subject to a lower mortality than the American, in order that the latter may coincide with the rate for active and invalid lives. Thus lives accepted a year ago, and not absolutely disabled, are treated as better than newly selected lives, and, generally, the longer the period since selection the better the mortality among those not totally disabled. Because of this palpable absurdity, Mr. Mead contends that the American mortality should apply to active lives entering at the youngest age, say 15 or 20, only; and that at all higher ages the rate of mortality among active lives shall be the same as among the survivors of entrants at the youngest age, and consequently lower than that of the American Table. This is theoretically more defensible, as the rate of mortality among active lives is then independent of the age at entry, but it does not comply with the requirement that the American Table shall apply at all ages and durations, as the aggregate mortality of all entrants

except those at the youngest age will be less than that of the American Table.

Mr. Pipe has pointed out that, by neglecting the above mentioned requirement, a life is treated as subject to different rates of mortality for the purposes of life and disability insurance respectively. I think, however, that a mortality table cannot properly be regarded as applying to a single life, but only to a group of lives. It is clear also that an individual may correctly be included in each of two or more groups, although different mortality tables apply to such groups. Thus a clergyman may properly be treated as subject to "clergy" mortality for the purposes of a clergy pension fund, and, at the same time, as subject to "insured" mortality for the purposes of life insurance—in the first case as belonging to a relatively small professional group, most of which would be included in the second and larger group. I am by no means certain, however, that practical convenience is not served in some directions by using the American Experience mortality in the manner in question.

Strict compliance with the foregoing requirement has generally been assumed to necessitate a separate mortality and disability table for each entry age. Mr. Hunter actually constructed such tables for quinary entry ages, and found that his premiums were almost identical with those when a single table, agreeing with the American for active lives at age 15 only, was used. Mr. Hunter assumed that disability benefits terminated upon survival as an active life to age 60 or 65, and would probably have found rather more difference had the limiting age been extended to 80.

It should be noted that when a single table is used, special formulas have to be employed so as to exclude from the calculations in respect of any given entry age x, the number of invalid lives l_x^{ii}, carried forward from previous ages, the calculations being based on the l_x^{aa} active lives only. The formulas make use of functions based on the American Table, but always with adjusting terms to exclude the effect of the assumed number at entry age x of invalid lives. The result is, that, except at a single commencing age, the American Table has no connection with the mortality of either active or invalid lives, or of both combined.

As illustrating the added complexity caused by using the American Table, it may be mentioned that Mr. Hunter gives formulas for reserves for waiver of premium benefits (*T. A. S. A.*, XII, 44)

which Mr. Mead considers to be based on a single table, and Mr. Pipe regards as based on the assumption of separate tables for each entry age, but which Mr. Hunter apparently was able to use for calculations from both sets of tables. A careful examination of the formulas shows that they do not involve any assumption as to single or separate age tables. All that is required is that the value of an annuity on an active life at the attained age shall be in accordance with the method used as for entry age.

All doubt as to proper reserves necessarily vanishes when pension fund methods are employed. The reserves are entirely independent of the regular life reserves, and may be kept so even when the sum insured is being paid by instalments, though as a matter of convenience, a single, though theoretically inexact, reserve would be held for such policies.

When the requirement that the American mortality shall equal that assumed for all ages and durations is to be strictly adhered to, it is not necessary to calculate separate tables for each age at entry, it being possible to avoid this by means of a special device which I shall now proceed to explain.

Taking a concrete example for the sake of clearness, we find that, by Mr. Hunter's table already referred to, at age 40 there are 77,625 active lives subject to "active" mortality and 481 invalid lives subject to "invalid" mortality. For all calculations by this table in respect of age at entry 40, only the 77,625 lives active at that age are taken into account, while to make the American Table apply the further 481 lives should be included as active lives, but subject to "invalid" mortality in order to keep the aggregate equal to the American Experience. Now if we were to calculate the premiums required to provide invalidity benefits for these 481 lives under the conditions named (i. e., they are to be treated as subject to the normal rates of invalidity, and to "invalid" mortality whether active or disabled), and added the premiums for the 77,625 active lives found in the ordinary way, the total would be the premiums for 78,106 lives subject to American mortality.

To apply this device, therefore, it is necessary to construct a table showing out of any convenient radix number, the numbers surviving as active lives and the number becoming disabled at each age, "invalid" mortality applying to both active and invalid lives, from which the ordinary tables of functions may be calculated. The required premium then will be,

$$\frac{l_x^{aa} \cdot P'_x + l_x^{ii} \cdot P''_x}{l_x},$$

where P'_x is the disability premium by the ordinary table and P''_x by the special table.

For the reserve after n years the formula would be,

$$\frac{l_{x+n}^{aa} \cdot {}_nV'_x + l_x^{ii} \cdot \frac{{}_{aa}l_{x+n}}{{}_{aa}l_x} \cdot {}_nV''_x}{l_{x+n}^{aa} + l_x^{ii} \cdot \frac{{}_{aa}l_{x+n}}{{}_{aa}l_x}},$$

where ${}_{aa}l_x$ represents the number of active lives by the special table.

One more subject requires mention. It has already been pointed out that the formulas of this paper imply aggregate rates of mortality and disability. No change in principle would be required, however, in order to apply select tables for both mortality of active and invalid lives, and for rates of disability. The tables would take the familiar form of select table functions, an extra column being required for each year for which selection is effective. If the period of selection were longer or shorter for disability than for mortality, the longer period would determine the extent of the tables. In practice, only the ultimate column need be completed in full, quinary intervals being sufficient for the balance. So far as concerns select mortality among the disabled, once the necessary values were tabulated, it would be just as easy to work with them as with the values of an aggregate table. An exception would occur in the reserves on disabled lives where temporary annuities would be required, for which complete tables would not ordinarily be constructed, the commutation columns being used instead. A similar difficulty would arise with the values of future premiums on active lives, for which the form,

$$P \cdot \frac{N_{+n} - N_{+t}}{D_{+n}}$$

would be used instead of

$$P \cdot \mathrm{a}_{+n:\overline{t-n}|}.$$

The question as to whether select tables should be used for these calculations is part of the larger question as to the desirability of

select tables entering into premium and reserve calculations, and may with advantage be left for inclusion in a discussion of the general subject.

Notation.

A word should be said about the notation. A general resemblance to Mr. George King's pension fund symbols will be recognized in the notation of the commutation columns. In some cases descriptive auxiliary symbols appeared unobtainable, and arbitrary ones had to be used. Instances are ${}^{r'}C_x$ and ${}^{rs}_{k}C_x$.

In the symbols for single and annual premiums, some attempt at consistency has been made. Taking the general form of the latter, as the more complicated, in the symbol ${}^{a}_{b}P_c$, a is used to describe the benefit, b the premium-paying term if other than for the maximum term of the contract, and c the term during which the benefit may come into possession. Thus ${}^{Wd}_{t}P_{\overline{y-x}|\,x:\overline{t}|}$ indicates the premium for *waiver* of *interest* (discount) for life, the premium being payable for t years, and the benefit continuing available during the joint existence of "x" and t years after $y-x$ years. Similarly in ${}^{W\overline{t}|}{}^{d}_{t}P_{\overline{y-x}|\,x:\overline{t}|}$ the waiver of interest lasts up to t years from original entry only.

In the symbol ${}^{E\overline{t}|k}A_{x\overline{t}|}$, the letter E is used to indicate that the benefit consists of a series of *Endowments,* as indicated in Section XXVI.

I must confess that I am not altogether satisfied with my scheme of notation, and trust a better one will be suggested.

Conclusion.

The final part of this paper is admittedly very incomplete. I have been compelled to leave many points of interest unnoticed, such as those arising from joint-life insurances, as it would be impossible to deal adequately with the entire subject within the limits of a single paper.

Legal Notes.

BY

WENDELL M. STRONG.

Dividend Estimates.—(O'Brien *vs.* Equitable Life Assurance Society of the United States, Supreme Court of Michigan, 138 N. W. Rep. 1086.) Like all dividend estimate cases decided by the highest court of a state, this case is important. The basis of the suit was a slip headed "Free Tontine Illustration Blank," and the suit was for an amount of surplus equal to that shown in the illustration. The illustration was delivered with the policy, and it was urged that this illustration constituted a part of the contract, reference being made to Timlin *vs.* Equitable Life Assurance Society. The court expressly refused to pass upon this question as it would not have affected the decision in the case. This decision was in favor of the company and depended upon the fact that, even if the illustration was admitted as a part of the contract, no promise of a definite amount of surplus was shown.

The wording of the policy and of the illustration with regard to surplus cannot be quoted in full, but the following extracts contain the statements which are of importance in determining whether a definite amount was promised or not. The policy referred to "the surplus then apportioned by the Society." The illustration preceded the figures under the various options with the statement, "Results at the end of the Tontine Period, on the basis explained on the other side of this sheet." The other side of the sheet referred to the surplus, in part as follows: "The surplus (popularly known as profits or dividends) which is a varying quantity depending on future experience." Again "the surplus is guaranteed also but its amount cannot be determined in advance." Again "while the results of the future must necessarily depend on the future. . . ." The court said in part:

"If it were conceded that Exhibit II (the illustration) was properly received in evidence, we are of opinion that the trial court, in determining that it constituted an agreement on the part of the defendant to pay a definite or guaranteed amount of surplus upon

this policy, incorrectly construed its language. We have here set out in full both Exhibits I (the policy) and II, that they may the more readily be construed together. A careful consideration of these instruments leads us to the conclusion that a reasonable person, reading them, would readily know that the amount of surplus is not fixed or definite, and could not be determined until the end of the tontine period, at which time it would be apportioned by the defendant, and that, if the experience of the defendant during the said tontine period should prove to be the same as its experience had been in the preceding twenty years, the surplus would amount to the estimate given, but otherwise not, and that the future alone could decide what the amount of the surplus would be, but that there would be some surplus was guaranteed."

The court also distinguished the language of the illustration in this case from that in the Timlin case, in that in the Timlin case the illustration used the word "estimate" only in the heading and in a statement that "guaranteed surplus is estimated at," whereas the option elected by Timlin and for which the suit was brought was stated " (3) Take a life annuity increasing annually beginning with $53.47."

Change of Beneficiary:—(Mutual Life Insurance Co. of New York *vs.* Lowther, Court of Appeals of Colorado, 126 Pac. Rep. 882.) The question was whether a change of beneficiary had been effected before the death of the insured. The policy gave the right to the insured to change the beneficiary by filing written notice thereof at the home office of the company, accompanied by the policy for suitable endorsement thereon. The time when such change should take effect was specified by the policy to be "upon the endorsement of the same on the policy by the company." The insured mailed the request, together with the policy, to the company but died before they reached the company. Thus nothing remained to be done except for the company to make the formal endorsement of the change on the policy. The insured had performed his part and, so far as he was concerned, the change was made and only the ministerial act of the company, which it had no option to refuse, remained to be done.

The court held, in line with most of the decisions bearing directly upon the subject, that the statement that the change should take effect upon its endorsement on the policy was not a condition

precedent and that the mere fact of the failure of the letter to reach the company before the insured's death was immaterial.

(Daugherty *vs.* Daugherty, Court of Appeals of Kentucky, 154 S. W. Rep. 9.) Practically the same question was involved in this case as in the above. The policy provided that the change of beneficiary should take effect upon the endorsement of the same on the policy by the company. The endorsement was made after the insured's death. The opinion of the court was in accordance with the above, that the fact that the change was not actually made until after the death of the insured did not invalidate it.

Bankruptcy—Ownership of Proceeds of Policy:—(Everett *vs.* Judson, U. S. Supreme Court. 228 U. S. 474.) The Bankruptcy Act contains the following specific provision in regard to insurance:

"Provided, that when any bankrupt shall have any insurance policy which has a cash surrender value payable to himself, his estate or personal representatives, he may, within thirty days after the cash surrender value has been ascertained and stated to the trustee by the company issuing the same, pay or secure to the trustee the sum so ascertained and stated, and continue to hold, own and carry such policy free from the claims of the creditors participating in the distribution of his estate under the bankruptcy proceedings, otherwise the policy shall pass to the trustee as assets."

The bankrupt held certain policies on his life payable to his estate. After the petition in bankruptcy had been filed, but before adjudication, he committed suicide. The trustee in bankruptcy claimed that the Bankruptcy Act vested the title to the property in the trustee as of the time of the adjudication, and that the death of the bankrupt before this, although after the filing of the petition, gave the trustee a right to the proceeds. The court held that the adjudication really related back to the time of filing the petition, saying:

"We think that the purpose of the law was to fix the line of cleavage with reference to the condition of the bankrupt estate as of the time at which the petition was filed and that the property which vests in the trustee at the time of adjudication is that which the bankrupt owned at the time of the filing of the petition. And it is as of that date that the surrender value of the insurance policies mentioned in section 70a should be ascertained."

This decision, therefore, gave to the trustee the surrender value as of the date of the petition, and held that the estate of the insured was entitled to the proceeds of the policy in excess of this.

Partridge *vs.* Andrews decided in the U. S. Circuit Court of Appeals and reviewed in "Legal Notes," May, 1912 (*T. A. S. A.*, XIII, 99), was decided on appeal in the Supreme Court of the U. S. (228 U. S. 479) on the same day as Everett *vs.* Judson and in a similar way. This reversed the decision of the Circuit Court of Appeals.

(Burlingham et al. *vs.* Crouse, Supreme Court of the U. S. 228 U. S. 459.) The policies in question were assigned to Crouse shortly before the petition in bankruptcy was filed. The insured died after the petition, and action was brought to set aside this assignment as a preference prohibited by the Bankruptcy Act. The court reviewed in the opinion, to some extent, the purpose of Congress in the provision regarding life insurance policies and also reviewed the nature of the life insurance contract as property, and came to the conclusion that: "It was the purpose of Congress to pass to the trustee that sum which was available to the bankrupt at the time of bankruptcy as a cash asset, otherwise to leave to the insured the benefit of his life insurance." The point was urged that in the provisions of the Act the cash surrender value was to be paid by the *bankrupt* when ascertained and the policies kept alive for his benefit, and that policies assigned absolutely, as in this case, would not come under the provision. The court rejected this argument, saying, "As we have construed the statute, its purpose was to vest the surrender value in the trustee for the benefit of the creditors and not otherwise to limit the bankrupt in dealing with his policy."

Materiality of Representation:—(Aetna Life Insurance Co. *vs.* Conway, Court of Appeals of Georgia, 75 S. E. Rep. 915; Fraternal Mystic Circle *vs.* Hansen, Court of Civil Appeals of Texas, 153 S. W. Rep. 351.) The question of what constitutes materiality of a representation is often the deciding factor in determining whether a policy can be avoided for misrepresentation. There has been considerable confusion upon this question and the statutes of two states provide that no false statement by the insured shall avoid the policy unless the matter falsified contributed to the death of the insured. This makes the question of

materiality depend upon the opinion of a jury, wholly untrained in medical matters, regarding a point on which frequently the ablest physician would be in doubt. Such a test seems unwise, for the reason that the determining factor can seldom be ascertained with certainty, and unfair, because the true test of whether the falsifying of facts by one party to a contract is material or not is whether it led the other party to enter into that contract when he would not otherwise have done so.

The present cases in their facts are commonplace, but are of note because of the clear statements of the doctrine of materiality in the following extracts from the opinion.

From Aetna Life Insurance Co. *vs.* Conway:

"The position assumed by counsel for the defendant in error is that the finding of the jury in favor of the plaintiff should not be disturbed, because they could have found that the insured did not have Bright's disease at the time the policy was issued, and that if the disease developed afterwards, and from causes which did not exist at the time the policy was issued, none of the representations made by the insured in reference to his previous illness and in reference to the physicians whom he had consulted and by whom he had been treated could have materially affected the risk. We do not believe that this is the exclusive test of the materiality of a representation. When an application for insurance is made, the attitude of the company is that if, in the opinion of its officers, the applicant is a desirable risk, his application will be accepted and the policy issued upon payment of the premium required. It is purely a matter of voluntary contract. The company is not bound to issue the policy, and may refuse to do so without giving any reason for its action. The question, therefore, is, if the applicant had dealt 'in the utmost good faith,' and disclosed the nature and extent of his previous illness, and disclosed the names of the physicians by whom he had been treated, would he still have been a desirable risk, and would the company have accepted him as such? 'In general, it may be said that the test in determining whether questions contained in an application for insurance are material is whether the knowledge or ignorance of the facts sought to be elicited thereby would materially influence the action of the insurer. . . .'"

From Fraternal Mystic Circle *vs.* Hansen:

"Any fact concerning the health, condition, or physical history of the applicant, which would naturally have influenced the insurance association in determining whether or not it would issue the certificate or grant his petition for reinstatement. . . . The

(lower) court, in its charge to the jury, defined the term as follows: 'You are charged that what is meant by "material to the risk assumed" is, Did Julius Hansen have any known disease on May 31, 1909, which caused, hastened, or contributed to his death?' and this charge, appellee contends, states the correct meaning of the words used by the statute. This was clearly error. The definition is far too narrow and limited in its scope. The Legislature did not mean to deprive insurance organizations of the right to freely exercise their judgment, in the light of complete information, as to whether they will or will not issue a policy of insurance upon the life of an applicant. . . . It made very plain to the applicant that it considered it important, in passing upon his application, to be advised of all the facts concerning his past, as well as his present physical condition."

Fraud of Beneficiary:—(Gamble *vs.* Metropolitan Life Insurance Co., Supreme Court of South Carolina, 75 S. E. Rep. 788.) It sometimes occurs that an application for insurance may be without fraud on the part of the insured and yet be tainted by fraud of the beneficiary. In the present case the insured was a woman who took out a policy in favor of her husband. The husband signed the application and paid the first premium. The insured, at the time of making the application, was suffering from a serious ailment of which, it was claimed, the beneficiary was cognizant and of which the insured was ignorant. The lower court had directed a verdict for the plaintiff on the ground that the pleadings alleged fraud on the part of the insured and not the beneficiary. The higher court reversed this and, in the brief opinion rendered, went beyond technicalities of procedure to the merits of the contention, speaking in part as follows:

"If he knew of his wife's fatal malady, and concealed it while participating in procuring the insurance for his own benefit, can he be allowed to say that his wife was ignorant of the imposition, when common honesty required him to inform her of her condition, if he wished to insure her life, so that she could speak the truth in her application? His request to the physician not to tell Mrs. Gamble of her condition was natural and commendable as a means of saving her from suffering, but it is not permissible that the plaintiff should pervert that sentiment by using it as a means of allowing his innocent wife to impose on an insurance company for his benefit. The plaintiff is now seeking his own interests, and he cannot have the benefit of his wife's lack of knowledge of her malady when it was known to him. He cannot be allowed to wilfully conceal, and then have the benefit of the concealment and the

imposition innocently practiced by another because of his concealment. If he wished her life insured for his benefit, it was his imperative duty to inform her so that she could speak the truth. The law will not allow him to deny that his wife knew that which common honesty required him to tell her before procuring insurance. As between the plaintiff and the insurance company, the law imposed upon him the duty of informing his wife of her condition to the end that she should not impose upon the insurance company in his behalf, and it will not allow him to say, in enforcing his claim, that he did not perform the duty."

The above does not rest upon the beneficiary's having signed the application, although it does depend upon his having participated in some way in the procuring of the insurance.

Forfeiture for Nonpayment of Premiums:—(Burton *vs.* Columbian National Life Insurance Co., 127 Pac. Rep. 1037.) This case is an illustration of the attempts which are made to twist the statements of the policy contract into a meaning never intended and which they will not reasonably bear. The policy was forfeited at the end of the first year for the nonpayment of premium. The insured died some six months later. Suit was brought, the claim being made upon the strength of the wording of the policy which promised to pay the amount of the insurance "less all indebtedness to the company on this policy together with any unpaid premium or portion of the premium for the then current policy year." It was contended that as the premiums were paid annually and as the contract expressly provided that if the premium for the then current policy year were unpaid it was to be deducted, this provision contemplated also the policy's remaining in force. The policy contained the statement that "Failure to pay any premium . . . when due will void this policy . . . and forfeit all premiums to the company, except as herein provided." It was claimed that the words "except as herein provided" referred to the above quoted provision of the policy interpreted as explained above. The court could not be convinced that any such interpretation of the contract was admissible and rendered a decision in favor of the company.

Delivery of Policy:—(Unterharnscheidt *vs.* Missouri State Life Insurance Co., Supreme Court of Iowa, 138 N. W. Rep. 459.) The premium was paid when the application for a policy was made.

Not long afterwards and before receipt of the policy, the insured fell sick and died. The policy at the time of the insured's illness and death had been sent from the home office and was in the office of the local agent. The interesting question in the case was whether the sending of the policy to the agent for delivery, the premium having already been paid, was sufficient to fulfill the condition of the application which read as follows:

"That the insurance hereby applied for shall not take effect unless the premium is paid and the policy delivered to and accepted by me during my lifetime and good health."

The court held that it did, treating the execution and the mailing of the contract as being equivalent to actual delivery, since nothing further remained to be done by the applicant. An extract from the opinion is:

"When the policy had been executed in conformity with the offer of the applicant, the minds of the parties had met; and the premium being paid, and nothing remaining for the applicant to do to perfect the contract, the execution by the company of the written evidence of that agreement and sending it, either direct to the applicant or to the local agent for his use according to the usual course of business, was all that the law or the terms stipulated in the application require to make a binding and enforceable contract."

Waiver of Payment of Dues within Required Time by Course of Dealing:—(The Royal Guardians, Appellants, and Clarke et al., Respondents. Cour du Banc Du Roi. Quebec. 21 Les Rapports Judiciaires, 540.) The certificate of membership in the order contained the condition that the member should comply with all "the laws, rules and requirements." The rule regarding assessments was that the assessments were due on the first day of each month and that "any member making default for thirty days to pay the same shall, *ipso facto,* be deemed suspended from all privileges of the order and his beneficiary certificate shall thereby lapse and become void." It was shown that, notwithstanding the above mentioned rule, the "financier" of the local lodge had been in the habit of collecting the dues payable on the first of one month by sending around to the members about the tenth of the following month; that such methods had been in use at least five years and were general to the members of the lodge and to other lodges; and that the dues were then accepted, not as reinstating suspended

members, but as the normal payment. Clarke died on the 7th of September, his dues payable the first of August not having been paid. They were, however, paid on the 8th of September, the day after his death. The court held that, in view of the general and continued practice, Clarke was a member in good standing and that the certificate was in force and the benefit thereunder payable.

Payment of Premium, Date of:—(Miner *vs.* Excelsior Life Assurance Co., Supreme Court of Alberta, 3 Alberta Law Rep. 249.) The defense was that the policy had not been delivered, and the first premium had not been paid during the life and continued good health of the insured. The application contained the condition, "This policy shall not take effect until the same be delivered and the first premium shall have been paid thereon during the life and continued good health of the insured." The policy was delivered to the insured on the 7th of April. On the 25th of April the insured called a physician who found him to be suffering from an acute attack of nephritis, and his death occurred on the 13th of July. The question was the time of the payment of premium. About the last of March, the general agents of the company had been notified to draw for the amount of the premium, and the money had been provided by the insured for the payment of the draft. The draft was actually made about the first of May, which was after the insured's health had become impaired. The court held that, as the insured had in fact provided for the premium before the delivery of the policy, it was practically paid, and that it was for this reason the policy was delivered, stating that: "the company's representative recognized that the premium was then virtually paid. The delivery of the policy and this virtual payment—so recognized by the company—were made at a time when, so far as appears, there was continued good health of the insured."

Amount Recoverable in Action by Insured for Return of Premiums Paid:—(Provident Savings Life Assurance Society of New York *vs.* Shearer, Court of Appeals of Kentucky, 151 S. W. Rep. 938.) This was a suit for return of premiums paid on a policy, with interest thereon. The decision of the lower court in favor of the plaintiff was reversed and the case remanded. The grounds for the decision and reversal are not of special interest to us, but the enlightened reasoning of the Court of Appeals upon the

proper amount of recovery, if the case should finally be decided against the company, is of importance. In some cases where a suit has been brought by the policyholder for the rescission of a policy contract and for the return of premiums with interest, this return has been granted as asked for, without recognition on the part of the court that the contract has been partially performed by the company, in that the company has carried the risk during the time the contract has been in force and would have paid the claim if death had occurred. The attitude of the Court of Appeals in the present case and of the Court of Civil Appeals of Texas in the case of Supreme Lodge Knights of Pythias *vs.* Neeley, discussed in the "Legal Notes" (*T. A. S. A.*, XII, 87), shows an increasing tendency to recognize the true facts and not allow the policyholder to enjoy the benefit of the insurance for years and then escape all payment for that benefit. An extract from the opinion relating to this is as follows:

"There is another feature of this case that needs to be noticed. If the judgment appealed from should be allowed to stand and the company be required to return to Shearer the premiums he paid, with interest thereon, the necessary effect would be to give Shearer the full benefit of the policy during the time it was in force without any cost to him. During this time Shearer was protected by the policy, and if he had died the company would have been compelled to have paid the amount it agreed to pay in the contract of insurance. It is therefore very plain that, if this contract is to be cancelled and rescinded, it should be done upon equitable terms. The insurance company should not be required to carry insurance on the life of Shearer for nearly five years without receiving any premium therefor, nor should Shearer be allowed to recover the premiums and yet not pay the insurance company anything for the insurance he had upon his life during the period covered by these premiums."

Damages for Unreasonable Delay in Acting on Application:—(Duffy *vs.* Bankers' Life Association of Des Moines, Supreme Court of Iowa, 139 N. W. Rep. 1087.) An application for insurance is an offer to enter into a contract which later the company accepts or rejects. It seems strange that delay in replying to an offer should be a ground of damages, whatever the cause, and at first sight that is what seems to be held in the present case. When considered in the light of the full facts, however, the decision is not so broad. Briefly, the facts are as follows: Mr. Duffy made

application for insurance and was examined, making the necessary payment by notes. He passed a good medical examination, his habits and finances were good, and his occupation, that of farmer, was not detrimental from an insurance standpoint. About a month after the application and examination he was drowned. The application, meanwhile, had not been acted on or even forwarded to the home office, but the home office had been notified that an examination had been made. Suit was brought for damages, not for a breach of contract but for tort (negligence), and in the lower court a verdict was directed for the company. The Supreme Court held that the issues should have been submitted to the jury. This seemed to be on the ground that the jury could have found from the evidence submitted that, had the application been acted upon, it would have been acceptable and Mr. Duffy would have been insured.

Mutual Benefit Insurance; Increase of Assessments:— (Green *vs.* Supreme Council of Royal Arcanum, Court of Appeals of New York, 100 N. E. Rep. 411.) In recent years it has been found necessary for many mutual benefit associations to increase their assessments, since the assessments originally provided for proved insufficient to meet the increased mortality that comes with the increase in age of such associations. Where the reservation of the right to increase assessments is not specifically reserved but exists, if at all, only under a general provision to comply with the laws of the order "now in force and which may be subsequently adopted," different jurisdictions are at variance as to whether such right exists. The present case states clearly the law of New York.

In the application, the member agreed to pay all dues and assessments for which he should become liable, "and to conform in all respects to the laws, rules and usages of the order now in force or which may hereafter be adopted by the same." The certificate issued to him contained the condition, "that the said member complies, in the future, with the laws, rules and regulations now governing the said council and fund, or that may hereafter be enacted by the Supreme Council to govern said council and fund."

The court held that these provisions were not sufficiently specific to allow increase of assessments, quoting as applying to this case from the opinion in a previous case as follows:

"It is quite easy for fraternal organizations, such as the defendant, if they deem the provisions for benefits to their members

tentative only and desire to have them subject to such modification as the business of the orders may require, to express that in the certificate. So, in the present case, if the certificate had provided that the payments therein specified should be subject to such modification as to amount, terms and conditions of payment and contingencies in which the same were payable as the endowment laws of the order from time to time might provide, the amendments would be applicable to existing members. But I think that nothing less explicit than this appearing in the certificate itself should be effectual for such a purpose. Fairness to persons joining the order required such plain dealing."

Power of Agent:—(Slocum *vs.* New York Life Insurance Co., U. S. Supreme Court, 228 U. S. 364.) On the last day of grace for payment of premium, and interest on a policy loan, the wife of the insured delivered to the agent a check for a portion of the amount, and received a blank "blue" note for the remainder, which, according to her testimony, the agent handed her saying that the note must be signed and returned. She told the agent that the insured was ill and it might be several days before the note could be returned, whereupon he said, "Mail it as soon as you can." The insured died without signing the note, and, as the policy provided for the payment of the premium annually, and plainly excluded payment by instalments, nothing was gained by the check handed the agent "unless what he did in that connection operated as a waiver of full and timely payment." The policy provided that only certain named officers had the power to modify the terms of the policy or to extend the time for paying a premium, but this had been modified by allowing agents to accept a part payment in cash, if accompanied by the form of note given out in this case. This practice was known to the insured and his wife who had taken advantage of it several times. Here, however, it was not a question of the "blue" note. The question was, rather, whether the agent had power to waive the full payment of the premium and either extend the time or accept a partial payment. The court held that he had not, speaking as follows:

"One who deals with an agent, knowing that he is clothed with a circumscribed authority and that his act transcends his powers, cannot hold his principal; and this is true whether the agent is a general or a special one, for a principal may limit the authority of one as well as of the other."

"Under the terms of the policy, as qualified by the practice of the

company, the agent was without authority to waive full and timely payment of the premium, save as he could adjust the payment conformably to the blue-note plan. His authority turned upon the giving of the note, which was a matter of real substance, and not of mere form, as is shown by the terms of the note, before quoted. Without it he could neither accept a partial payment nor extend the time for paying the balance. No note was given, and so no waiver resulted from his acts. The insured and his wife could not reasonably have understood it otherwise, for they knew the terms of the policy and were familiar with the qualifying practice."

One of the great difficulties that life insurance companies have to meet is the tendency of the courts to extend the powers of agents employed by the companies as solicitors beyond the power intended to be delegated them by the company and beyond the power which they are held out by the company as possessing. A decision by the Supreme Court of the United States as to the limits of the power of an agent is particularly valuable under such circumstances.

Limitation of Amount of Insurance on an Infant:—(Flynn *vs.* Prudential Life Insurance Co., Court of Appeals of New York, 100 N. E. Rep. 794.) The New York Statute regarding the amount of insurance which may be taken on an infant read, in part, as follows, ". . . a person liable for the support of a child . . . may take a yearly renewable term policy of insurance thereon, the amount payable under which may be made to increase with advancing age, and which shall not exceed the sums specified in the following table, the ages therein specified being the age at time of death." Policies were held both in the Metropolitan and in the Prudential, the insurance in the Metropolitan being for practically the full amount allowed by the statute. The Prudential contested the claim upon the ground that the Metropolitan insurance of the full amount that the law allowed had been paid. The Appellate Division held that under the recognized rules of construction the statute did not limit the total amount of insurance but the amount of any one policy. ("Legal Notes," *T. A. S. A.*, XII, 269.) The Court of Appeals, construing the statute with regard to its manifest intent and meaning, held that it was the total amount of insurance which was limited and not merely the amount of a single policy, thus reversing the Appellate Division. An extract from the opinion is:

"What possible ground can be suggested by the most ingenious mind for the requirement that unlimited insurance should be ef-

fected only by several policies, each of which should not exceed the sum prescribed by the statute? The evil sought to be guarded against by the statute was the neglect or actual maltreatment of infants of tender years by those liable for their support for the purpose of making pecuniary profit out of the infant's death. This result could be effected by limiting the amount of insurance permitted on the infant's life, not at all by limiting the amount of a single policy, if other policies were to be permitted."

Cancellation of Policy by Consent of Insured where Right to Change Beneficiary is Reserved:—(Indiana National Life Insurance Co. *vs.* McGinnis, Supreme Court of Indiana, 101 N. E. Rep. 289.) The company claimed fraud in the procurement of the policy, and by agreement between the insured and the company the policy was cancelled prior to insured's death and the premium paid, with interest, was returned to the insured. After the death of the insured, the beneficiary brought suit, claiming that, even though the policy contained a clause reserving to the insured the right to change the beneficiary, the beneficiary had such an interest in the policy from its issuance that it was beyond the power of the insurer and the insured to annul the policy without the consent of the beneficiary except by making such change first.

The court held that the taking of the contract gave a vested right to the beneficiary which could be divested only by the exercise, strictly according to the terms of the contract, of the right to change the beneficiary. The beneficiary, having then a legal interest in the policy at the time of its annulment in which she had not joined, could not be deprived of that right by such annulment while she remained beneficiary.

Right to Withdraw Application:—(Wheelock *vs.* Clark, Supreme Court of Wyoming, 131 Pac. Rep. 35.) Clark made application for insurance, and at the same time executed and delivered a note for the amount of the first premium and took a receipt for the same, which receipt stated that the amount collected should be recognized as the first annual premium, "provided the application is approved by the company at its home office, and in that event the insurance applied for will be in force from the date of the medical examination." Later, but before any action upon the application, he notified the company by wire that he wished to withdraw the application and confirmed the same by letter. The court held that

Clark had a right to do this, and to recover any payment he had made, saying, "we have no hesitation in holding that no contract of insurance was entered into between Clark and the company." The opinion of the court further showed that it held the application, until accepted, to be a mere proposal to enter into a contract, and that no contractual relations existed between the company and the applicant until such acceptance.

Deferred Dividend Contract; Claim of Right to Benefit from Surplus before End of Dividend Period:—(Jefferson et al. *vs.* New York Life Insurance Co., Court of Appeals of Kentucky, 152 S. W. Rep. 780.) The policy involved was a deferred dividend policy. It had lapsed, and under its provisions was continued as extended insurance, the term of which expired some months before the death of the insured took place. Notwithstanding the fact that the policy provided it should not receive any dividend unless in force at the end of the fifteen-year dividend period, it was claimed that dividends accumulated upon the policy but not apportioned to it were sufficient to carry the extended insurance beyond the date of death of the insured, and should be so applied to carry the extended insurance.

This case is of particular importance because it practically overrules the well-known Spinks case, decided in the same state some years ago. The conclusion in the Spinks case was that the insurance was in force until after the insured's death, this result being reached by giving to the words "dividend addition" a wrong meaning and, through this, apportioning surplus to the policy before the end of the dividend period, although the policy contract provided that it was to be apportioned only at the end of the period. In the present case it was contended likewise that surplus should be apportioned notwithstanding the provisions of the policy contract. The court, however, upheld the terms of the contract that no surplus was attributable to the policy at the time of lapse; stating its reason therefor very clearly as follows:

"Most insurance companies write different kinds of insurance; that is policies upon different plans. Each kind has its own peculiar, distinguishing features. . . . The rights of the insured vary under the different forms or classes of insurance. These facts are taken into account by the contracting parties, and in most, if not all, cases, exercise a controlling influence in making the contract.

A deferred dividend policy cannot be treated by either the company or the assured, as though it were an annual dividend policy, any more than an ordinary life policy could be held to be subject to the provisions of a term policy. It is altogether probable that in accepting a deferred dividend policy the assured secured rights which he could not have secured under an annual dividend policy; and undoubtedly corresponding rights accrue to the company, one of which was that providing that, unless the contract was kept in force by the insured until the dividend period arrived, it would not be accountable to the assured for any dividends thereon."

In the course of the opinion, particular attention was given to the meaning of the words "dividend addition," and their true meaning, in contrast to the wrong meaning assigned them in the Spinks case, was clearly and forcibly stated as follows:

"The term 'dividend addition' is perhaps more frequently misapplied than any other term used in connection with insurance. It means something added to the policy in the shape or form of paid-up insurance. The term originated in this way: When a dividend was apportioned to a policy and not drawn by the policyholder in cash, or used by him in reducing his premium payments, the company would issue to the insured a paid-up policy, payable at his death, for such an amount as the dividend, applied as a single premium, would buy; and this little policy, fully paid-up, was appended to the original policy, and was called a 'dividend addition' to the policy. It, in no sense, represents unapportioned assets, or surplus, of the company, but has reference solely to paid-up insurance."

Misstatement of Age:—(Metropolitan Life Insurance Co. *vs.* Stiewing, St. Louis Court of Appeals, 155 S. W. Rep. 900.) The policy contained a clause that the amount payable thereunder should be the insurance which the actual premium paid would have purchased at the true age of the insured. It was claimed by the company that the insured was seventy-two years old at the time the policy was taken instead of fifty-three, the age stated in the application, and that, consequently, the amount of insurance was really the amount paid for at age seventy-two by the premium received. The court held against this on the basis of the Missouri statute reading, "No misrepresentation made in obtaining or securing a policy of insurance on the life or lives of any person or persons, citizens of this state, shall be deemed material, or render the policy void, unless the matter misrepresented shall have actually contrib-

uted to the contingency or event on which the policy is to become due and payable, and whether it so contributed in any case shall be a question for the jury." Special emphasis was laid by the court on the word "material," and it seemed to be considered that a diminishing of the amount of the policy to that paid for at the true age was making a misrepresentation "material" in the sense of the statute.

It seems as if the above were not the correct construction of the statute. The policy provided for an amount of insurance dependent upon the true age, stating what that amount was if the age were fifty-three, but did not promise to pay the amount so stated unless the age were fifty-three. The amount of insurance actually promised was to be equal to, greater than, or less than, the amount stated for age fifty-three according as the age at issue was equal to, less than, or greater than, fifty-three. Thus the policy provided, not that a misrepresentation should be material, but that the true age should equitably determine the amount of insurance. The statute evidently was not intended to defeat a contract providing that the amount payable should be the amount which the money received by the company paid for at the true age of the insured, but rather to apply to an attempt to avoid a policy.

The similar case of Metropolitan *vs.* Burns was reviewed in "Legal Notes" for May 1910 (*T. A. S. A.*, XI, 337).

Relation of Company to Insured; Accounting:—(Equitable Life Assurance Society of the United States *vs.* Weil, Supreme Court of Mississippi, 60 South. Rep. 133.) The case was that of a bill in equity either for an accounting or for discovery and relief, the court did not determine which, to which the company demurred. At the end of the tontine period, Weil accepted under protest the amount tendered to him and filed the bill charging fraud. No facts were alleged to support this charge, and the court held that the simple charge was not sufficient.

It was urged that the relations between the insurer and the insured were of a fiduciary character and that the former was in the position of a trustee. The court held that they were not but rather that "the relation between the policyholder and the company was one of contract measured by the terms of the policy."

A further interesting point in the decision is that the court held

that; "In the interpretation of this contract the decisions of the New York courts should be followed."

THE POLICY CONTRACT; WHAT IS INCLUDED THEREIN?—(Penn Mutual Life Insurance Co. *vs.* Gordon, Supreme Court of Mississippi, 61 South Rep. 311.) The policy in this case was signed at the end of the first page, and the second and fourth pages were made a part of the contract by the following reference upon the first page: "The extended insurance, paid-up insurance and cash surrender value privileges, benefits and conditions stated on the second and fourth pages hereof form a part of this policy as fully as though recited at length over the signatures hereto affixed." It was claimed that the condition regarding suicide on the second page of the policy was not a part of the policy, the argument being made that the terms "benefits and conditions" in the clause quoted above were limited by the preceding words, "extended insurance, paid-up insurance and cash surrender value." The court held that the clause in question was a part of the policy. An extract from the opinion is:

"A complete inspection of the entire contract of insurance, as well as a careful consideration of the reference clauses, leads us to the conclusion that appellee is wrong in her contention. We believe that the proper reading of the clause will include, by its reference, not only the provisions as to extended insurance, paid-up insurance and cash surrender value privileges, which provisions are in a large measure tables set out in the instrument, but also include the benefits and conditions set forth in the instrument. We not only consider this the proper reading of the clause, but it seems clear to us to have been the intention of the parties in making the contract. It also seems a reasonable construction, evidencing the plain spirit of the instrument, gathered in one view from the whole."

Abstract of the Discussion of Papers Read at the Previous Meeting.

THE EFFECT ON SELECT TABLES OF A VARIATION IN THE RATES OF MORTALITY TO WHICH THE LIVES INVOLVED ARE SUBJECT—PERCY C. H. PAPPS.

VOL. XIII, PAGE 211.

WRITTEN DISCUSSION.

MR. JOHNSTON:

Mr. Papps's paper concerning the effect on select tables of a variation in the rates of mortality is an interesting theoretical treatment of the subject. In his paper, however, he treats with a special case of the problem in that his calculations are based upon a progressive improvement in the rate of mortality. Upon general reasoning it is self-evident that, on the assumptions made, a select table based upon the experience as a whole would be more favorable than would be the case if it were based upon the business issued at the beginning of the period, and less favorable than it would have been under the business issued at the end of the period. It is also evident that the degree of variation would depend to a considerable extent upon the relative amount of business issued from year to year.

There is one slight inconsistency in his assumption, at the top of page 212, where he states "If it be assumed that an equal amount of insurance, number of policies, or number of lives be placed on the books each year, and that the rate of cessation does not vary, then $l^{1}_{x+1} = l^{2}_{x+2} = \text{etc.}$" With a decreasing death rate this condition could not obtain, for with a decreasing death rate the number living at attained age x out of the more recent entrants will be greater than out of an equal number who entered earlier in the period of observation. The error, however, is comparatively slight.

From a practical standpoint it is very doubtful whether we would ever encounter such a special case as assumed; that is, where there is a continued progressive improvement in the rates of mortality. While a decided improvement might be shown over a reasonably long period, the variation from year to year would be more or less erratic. It is also to be pointed out that the result to which Mr. Papps has drawn attention can arise only where a mortality table

has been constructed in the special fashion assumed; that is, where the experience of the first policy year extends over twenty years, while the experience of the twentieth year is that of the last year only of the observations. If we take the case of the existing $O^{[M]}$ Table and suppose that the conditions assumed by Mr. Papps held during the period 1863 to 1893, it is interesting to note what would have been the probable effect. In this investigation the experience of the first policy year was that of the issues of 1863 to 1892, inclusive; of the second policy year that of the issues of 1862 to 1891, inclusive; of the third policy year that of the issues of 1861 to 1890, inclusive, and so on. If now we make the assumption that the distribution of the exposed to risk for any one policy year was chronologically the same as for any other policy year—an asumption departing a very little from the fact—we shall arrive at the conclusion that the degree of selection shown in the $O^{[M]}$ Table is wholly unaffected by the existence of a generally decreasing mortality over the period of the observations. This will be evident when we remember that the same proportion of the total exposures of each policy year is allotted to any given year of observation; so that an identical series of percentages of the mortality of the first year of observation applies to each policy year.

In symbols, if the proportions of the total exposures for the successive years of observation are

$$K_1, K_2, K_3, \cdots, K_{30}$$

for all policy years alike, where

$$K_1 + K_2 + K_3 + \cdots + K_{30} = 1,$$

the calculated mortality rate will be

$$\begin{aligned} &K_1 q_{[x]+n} + K_2 r q_{[x]+n} + K_3 r^2 q_{[x]+n} + \cdots + K_{30} r^{29} q_{[x]+n} \\ &\quad = q_{[x]+n}\{K_1 + K_2 r + K_3 r^2 + \cdots + K_{30} r^{29}\} \\ &\quad = q_{[x]+n} \times \text{a constant} \end{aligned}$$

for all values of n.

It thus appears that the actual mortality at all durations is a constant percentage of what would have been exhibited had the general mortality remained constant over the period of observation, whence it follows that the relative effect of selection is the same for constant as for decreasing mortality.

In the expression

$$q_{[x]+n}\{K_1 + K_2 r + K_3 r^2 + \cdots + K_{30} r^{29}\}$$

it is obvious that for r, r^2, r^3, $\cdots$, r^{29} we may substitute any arbitrary series, and the factor within the brackets will remain a constant for all values of n. It follows that fluctuations in any direc-

tion, erratic or uniform, in the mortality of the various years of observation leave unaffected the relative force of selection, provided only that the chronological distribution of the total exposures is the same for all policy years. This condition will practically obtain when select tables are constructed by a proper method—such as that of the $O^{[M]}$ Table.

It might be well to draw attention to several points in the text of Mr. Papps's paper. In the second paragraph, on page 211, explaining the symbol $d^{t/n}$, n is used to represent the nth calendar year, but at first it is not clear as to whether t refers to the policy year or year of investigation. From the text, however, it develops that it represents year of investigation.

On page 211, q is given as "the probability of dying according to some standard table," while on the next page it is used to represent "the mortality in the first year of the investigation."

In Table A, the heading $q_{[x]+10}$ should be q_{x+10}, this rate holding for all policy years from the 11th onward, for a constant age attained. Tables A^1, A^2 and A^3 have not been continued beyond the 11th policy year, but by multiplying the constant values of q_{x+n}, where $n > q$, by the decreasing series of M_n these tables might have been continued for twenty years, and would have shown a steadily decreasing rate of mortality by attained age, after the 11th policy year.

MR. FILE:

In mortality experiences hitherto deduced, it has not been found practicable to do other than to group together results for the whole period under investigation, regardless of the fact that usually there is a varying mortality more or less marked, but in modern experiences, at least, showing a decided improvement from year to year. Mr. Papps's paper is an attempt to measure or rather to discover general indications as to how the true mortality curve is affected by the grouping together of what may be called, owing to the changing conditions of life of those included in the mortality experience group, heterogeneous data. The author takes an example assuming that the improvement in mortality each succeeding year under investigation can be measured by a geometrical progression and concludes that in the resultant mortality table based on the whole period under investigation the full effects of selection are not shown, that there really exist greater effects of selection than such a table exhibits. The author adopts mortality ratios as a means of measuring the effects of selection. Similar results might have been shown by simply accumulating the difference in the rates of mortality for ten years for corresponding ages at entry under the standard table and the several final tables A^1, A^2 and A^3. The accumulated difference between the standard table and table A^1 is greater than that between A^1 and A^2, and the latter in turn is greater than that between A^2 and A^3, indicating that an increasing

improvement in the rates of mortality experienced does not show a corresponding increasing accumulated decrease in the rates of mortality in the final table. In other words, each of the final tables exhibits less effects of selection than does the standard table, and amongst the final tables themselves similar results are to be found. Of course, neither of these two methods is accurate, but they afford rough means of determining the general trend of the effects produced.

But while it seems clear that the final tables exhibit less effects of selection than does the original standard table, I cannot, as yet, agree with Mr. Papps that the selection shown is less than really exists. It seems to me that the results shown are in accord with what general considerations would lead us to expect. The improvement in mortality noted to-day, is due to several causes, amongst which are improved sanitary conditions, improved social conditions and developments in medicine. Now when an insurance company selects its policyholders, it pays due regard to all these conditions, and selection comes into play. Consider, for example, the case of inhabitants of unhealthy districts. Selection is exercised by the companies on this account, but if those sections of the country, at present unhealthy, were made healthy, there would be no further need of selection from this cause and it would cease to exist. Similarly if a cure for tuberculosis were discovered, there would be less reason for selection being exercised and consequently there would be less selection. All improvements then in social, sanitary or medical science reduce the field of selection, and, as a result, less selection exists. It follows then that in constructing a mortality table covering a period of time during which there has been a continuous improvement in the health of the general population, we would expect a table reflecting the mortality of the last few years to show less selection than one reflecting the mortality of the early years, and, consequently, a table that embraced the experience of the whole period in question should show less effects of selection than some table which represented the mortality experienced at a period of time immediately preceding the period covered by the investigation. In my opinion such a table should show less effects of selection, and as from general considerations there is no reason to suppose that the full effects of selection are not shown, I must take exception to Mr. Papps's conclusion on this point. Whether or not this apparent anomaly is due to the employment of mortality ratios as a means of measuring the effects of selection, I am unable to say without further investigation, and it is to be hoped that Mr. Papps may find time to pursue his investigations further in the hope that additional light may be thrown on this particular point.

Before concluding, I should like to refer to the desirability for uniformity in the notation used by members in presenting papers. There is apt to be confusion caused by employment of symbols with

a different meaning from that usually understood. Thus in this paper, Mr. Papps has used the symbol "Q" in quite a different meaning from that adopted by the International Congress of Actuaries. This practice tends to make it more difficult to follow the argument of a paper, and it seems to me that more uniformity in practice is highly desirable.

MR. PAPPS:

(AUTHOR'S REVIEW OF DISCUSSIONS.)

I wish to thank those who have been good enough to discuss the paper which I presented at the last meeting. I think it is only fair to make the statement that this article was originally prepared as an experiment to endeavor to account in some measure for the fact that the M. A. Table, to be used as a standard in the Medico-Actuarial Investigation, showed a period of selection lasting for only five years. This result was so unexpected that I endeavored to ascertain whether any change in the general mortality of the community over the period of years covered by the table would have accounted for the apparently short period of selection.

Referring to the last paragraph of Mr. File's criticism, I wish to say that while I agree entirely with Mr. File that it is very desirable to adhere to uniform notation, I am of the opinion that, especially with such a paper as I had the privilege of presenting, which is considerably out of the ordinary run of papers, the adoption of a notation applicable to that paper alone is not objectionable. Furthermore, it is almost impossible to find special symbols which have not already had some meaning attached. Mr. Johnston refers to the use of the symbol "q," which is taken to represent the probability of dying according to some standard table, and afterwards is used to represent the mortality in the first year of the investigation. The reason for what appears to Mr. Johnston to be a double use of the same symbol is that it was the intention that the mortality in the first year of the investigation should be represented by the "q" of the standard table.

A SUGGESTION FOR THE USE OF STATISTICS BASED UPON EUROPEAN EXPERIENCE WITH WORKMEN'S COMPENSATION IN ARRIVING AT PREMIUM RATES FOR INSURANCE COVERING THIS RISK IN THE UNITED STATES—

A. H. MOWBRAY.

VOL. XIII, PAGE 221.

WRITTEN DISCUSSION.

MR. FORBES:

Mr. Mowbray's very interesting paper is an effort to point out in concrete form a method for using European data, collected by the United States Bureau of Labor, to solve some of the problems discussed by the Society in a previous paper by Mr. Watt. Mr. Mowbray realizes that the proposition is a very complex one, and that the data derived from German and Austrian statistics are not in the best form for the purpose, and are, at best, a very rough guide in making rates for American risks. The companies, however, are indebted to Mr. Mowbray for placing in actuarial form an outline of a solution of the problem which is the most vital one now before them, and which they hope shortly to work out in a practical manner. I refer specifically to the organization of a department of the Workmen's Compensation Service Bureau for the rating of risks on the so-called merit system. The demand of the public for differential ratings is acknowledged, and such ratings must necessarily be based upon inspection and accident experience.

The organization of an inspection department of the Workmen's Compensation Service Bureau is actively under way. The obstacles are: first, the selection of a proper manager for the Bureau as a whole, and a chief inspector who will be both a practical underwriter and a practical engineer, and second, provision for the cost of such a bureau which apparently must be borne by the so-called "Bureau Companies" without assistance from the non-board companies. It is perhaps unnecessary to state that at least one third of the responsible companies doing business in New York State are not members of the Bureau, and that such companies are apparently not willing to incur any portion of the expense of inspection and rating of risks on a broad and comprehensive plan which will be fair to all parties at interest. The Society will possibly recall that the same problem was met by the fire insurance companies in their early organization. The New York Fire Insurance Exchange, in its present solid and efficient form did not always exist. The expense of main-

taining such an organization must be borne by the companies members of it, and the companies, in turn, must have some reasonable certainty that the rates fixed by the organization can be secured; otherwise it appears that the work done by them will be destroyed by the outside companies who are ready to issue policies without regard to rates established by the Bureau, and oft-times without knowledge of the fundamental principles and facts involved in such ratings.

Mr. Mowbray has outlined a scale of rating based upon an ideal risk, with penalties added to the basic rate for defects ascertained by inspection. For practical purposes, however, the American manufacturer has been educated to a series of discounts for improvements, rather than a series of penalties for lack of safeguards. The American public loves a bargain. It is a curious fact that although the Bureau has a system of advanced rates higher than the manual, which in theory is fully as complete as the system of reduced rates, that a hundred reduced rates are issued for every advanced rate. Mr. Mowbray's plan, however, is readily adaptable to the system of rating in which the manual rate applies to the unguarded risk, and safeguards found by inspection are made the basis of discounts. This means a much higher system of manual rates, but leads to the same result, and pleases the American public much better.

The practical difficulties of an inspection service, aside from the cost, are very great, owing to the lack of trained liability inspectors. The attention of recent graduates of engineering schools, however, is being drawn to this subject. The results obtained from these sources in the past year, to the writer's knowledge, are very encouraging, and we believe that liability inspectors of the future will, in a large percentage, be recruited from the graduates of engineering schools rather than from the ranks of mechanics and boiler inspectors. The companies realize that inspection as well as statistics must form a vital part of the underwriting system in the future.

Assuming that satisfactory inspection service can be established, and that a bureau of sufficient strength to enforce, in the main, its rating principles can be maintained; assuming, furthermore, that the companies are given a fair chance by the various legislatures to work out a fair system of merit rating, and that the adaptation of the various compensation laws, no two of which are alike in application or in benefits, is a matter which can be satisfactorily worked out by the statisticians; there remains to be ascertained a sufficient amount of available statistics to form a fundamental groundwork for the elaboration of a rating system. From a purely actuarial standpoint, the neatly compiled German and Austrian statistics appear to be ready at hand, and the disinclination of the practical underwriter to avail himself of this information is oft-times ascribed either to ignorance or to obstinacy. The real objection to the use of these figures is perhaps based upon the vague but substantial fear that results worked out on European statistics will produce rates which applied

to American conditions will bankrupt every company that attempts to use them. It will be very difficult to show why woodworking rates which vary from $5.60 to $1.85 were considered necessary by the underwriters compiling the Illinois Manual, in the face of Mr. Mowbray's conclusion that a basic rate of 79½c. can be used, plus the necessary loading for medical aid and expense. A very casual examination of the records of the United States Bureau of Labor reveals that the number and severity of the accidents which arise from American industry are a frightful commentary upon the carelessness and haste of our American industrial system as compared with those upon the Continent. The American factory is still upon the frontier of industrial development. It is a tremendously efficient, but also a tremendously wasteful system. Its raw materials, both concrete and human, are not comparable with German materials. The industry which descends from father to son, and employs as apprentices the sons of its laborers, is a rarity in this country. Our mills digest the crudest labor of Southern and Central Europe. They educate the Slav to three centuries of world-progress in a decade. The tiller of the soil, whose most complicated machine has been a plow, is asked, in language which he does not understand, to make delicate adjustments of a machine which grinds out his life in a moment of fatigue. Our manufacturers are not necessarily lacking in humane sentiments, but they have not yet had time to adapt themselves to conditions. It is a well known underwriting fact that a rapidly growing factory is worth nearly double the rate of premium that is required to carry a stable institution. Such an economic development as is evidenced by the automobile industry in Detroit, where factories grow from a $50,000 payroll in one year to $2,500,000 in one year, is practically unheard of in Germany. The American underwriter is somewhat in the position of a navigator who sees a storm brewing, shortens sail, and hopes for the best.

Liability underwriting, however, in the past has not been, as often claimed, entirely a matter of guesswork. The liability companies have been consistently compiling statistics through the old Liability Conference, through the statistical bureau maintained by the companies surviving that conference, and through the statistical departments of the leading companies. They have available a large fund of information tending to show the comparative hazards of various American industries. The experience derived from these sources under liability laws is a more accurate guide to workmen's compensation ratings than might be supposed. While settlements have theoretically been made upon legal liability, the practice of all companies, particularly for the past five years, has been to settle on the basis of injury without much regard to the question of liability involved. This business practice has been necessitated by the unscrupulous ambulance chaser who can introduce testimony which will take almost any case of serious injury to a jury, who will pass upon the matter by the rule of thumb standard of what ought to be

the law rather than what is the law. Judges are more and more loath to interfere with the findings of juries in negligence cases, and recent employers' liability acts leave so few valid defences that the horrible examples of injustice brought out before state commissions are the exception rather than the rule. Practically any workman, under any liability act in force, can secure a fair settlement for serious injury, provided his bargain with his lawyer does not deprive him of the result of such a settlement. The companies also have available a much more valuable but smaller mass of workmen's wage statistics, in which settlements are based upon actual periods of disability under a fixed scale of compensation. The adaptation of these statistics to compensation laws is strictly an actuarial problem. These statistics have been made the groundwork of the various compensation manuals of rates which have been published in the past three years. While the classes of business in which workmen's wage policies are carried in sufficient volume to establish adequate ratios are limited, it is possible to derive from these figures basic rates in certain leading classifications. The liability experience available, as pointed out above, shows the relative hazard between practically all classifications, and the application of the ratios of hazard derived from liability statistics, in working out the interpolations necessary between classifications based upon workmen's wage statistics has produced the present schedule of workmen's compensation ratings.

Contrary to the expectation in Mr. Mowbray's article, the practical difficulty owing to lack of data which has confronted the companies has been the proper allowance for the amount payable in the first and second weeks. Almost all workmen's wage policies in the past have made payment for injuries exceeding three days. A number of the compensation acts eliminate entirely the payment for the first two weeks. It has been found in practice, particularly as a result of the Massachusetts experience for the first six months of the Massachusetts Act, that the companies overestimated the amount which would be paid for minor claims, and that a reduction in rates could safely be made on account of the saving which was derived from the omission of any payment except medical attendance in the first two weeks. It has been found, on the other hand, that the amount estimated for medical attention has not been adequate as a whole. These adjustments, however, are matters upon which a comparatively short experience produces a sufficient volume to give a reliable index, and a general reduction of rates in Massachusetts of 25 per cent. was made effective by all companies on February 15 last. It may be assumed, therefore, that rates in general satisfactory have been roughly compiled, and that such rates are perhaps more reliable than a similar system derived from European statistics.

The further problem—to establish a method of differentiation between risks based upon inspection—is still in its infancy. The companies are taking steps at the present time to re-compile their

available data on the basis of class of accident. Such data will serve as a basis for differentiations based upon inspections. The conviction of the writer is that the ratios derived, as given in Table 4 of Mr. Mowbray's article, will be found to show considerable variation when based upon American statistics. For instance, the number and percentage of cost of elevator accidents in this country is, in the writer's belief, much higher than given in Table 4. Again, the hazard from burning buildings under compensation acts calling for unlimited liability, such as provided by the Massachusetts law, is bound to be a very material part of the future exposure. It is estimated that under the present law, the liability of the insurance companies for an accident similar to that of the Brockton Shoe Company boiler explosion a few years ago, would have been at least half a million dollars. Such a loss would necessarily affect rates based upon physical conditions to a very material extent. It serves as a striking illustration of the difference in such percentages which must be made under the different compensation acts. For instance, in New Jersey the total limit for compensation assumed by the insurance company has been $10,000. In Massachusetts the law provides that the insurance company shall assume the entire hazard. An accident similar to the Brockton disaster stated above would, therefore, cost the companies half a million dollars in Massachusetts, and only $10,000 in New Jersey, and the percentage of penalty charged for bad boiler conditions, or the differential allowed for good boiler conditions, would vary materially between these two states.

We believe that sufficient has been said to point out to the Society both the extreme difficulty of the subject, and the efforts which the insurance companies are making to establish a system which will meet with the approval of the public. There is no business that is more peculiarly sensitive to public sentiment than the insurance business. There is perhaps no business which appeals more to the agitator of socialistic principles as a fit subject for special action. The ease with which a state insurance scheme can be started, and the failure on the part of the public to appreciate the difficulties of the problem, render the insurance companies peculiarly liable to attack.

The salvation of the life insurance business, and one of the principal reasons for its rehabilitation in the public estimation, has been the mutual system. Some feeble efforts have been made in the casualty business to work out a practical mutual system of stock insurance. The weakness of the purely mutual companies is their liability to assessment. The life insurance companies, by their more accurate knowledge of their problem, by their large accumulations of reserves, and the general stability of their policies, have been able to work out a mutual system which does away with assessments. The average casualty policy is written for a year, which is too short a time to estimate profits. There is still too much left

to chance. We believe, however, that a practical mutual scheme, based upon the carrying of a risk of reasonable size for a period of at least five years, can be worked out, and that the ratings of the future will be based partly upon a discount given in advance for physical improvements, and partly upon a dividend declared under certain restricted conditions based upon the psychological factors which are evidenced by freedom from accident in the particular plant which is the basis of the adjustment, and also in part upon the class experience as a whole, as evidenced either by all of the companies, or by the particular company.

MR. PAPPS:

It was with some considerable hesitation that I accepted the invitation to prepare a written discussion of Mr. Mowbray's paper, for I have had no occasion or leisure to make a study of workmen's compensation insurance. At the same time I am glad of the opportunity of referring to one or two points which occurred to me in reading Mr. Mowbray's interesting paper.

On page 221 Mr. Mowbray quotes from a report relating to the Remington Typewriter works as follows: "During the five year period ending December 31, 1911, the number of employees increased 66 per cent., the number of machines increased 25 per cent., and accidents of all kinds per one thousand employees decreased 46 per cent." The natural conclusion which might be drawn from this statement is that the rate of accident has decreased 46 per cent. If it be assumed that the accidents are caused only by machines, it will be found that the apparently large decrease in the rate of accident is largely due to the fact that the number of employees increased in greater ratio than the number of machines. The rate of accident per machine may be found as follows:—

Let N = number of employees in 1911 = $(1+b)n$ where
n = number of employees in 1906,
M = number of machines in 1911 = $(1+c)m$ where
m = number of machines in 1906,
A = number of accidents in 1911
a = number of accidents in 1906,
R = number of accidents per machine in 1911 = $(1+d)r$ where
r = number of accidents per machine in 1906.

Then $A = R \cdot M$, and number of accidents per employee is $\frac{R \cdot M}{N}$ in 1911, or $\frac{r \cdot m}{n}$ in 1906. The decrease in accidents per employee is

$$\frac{r \cdot m}{n} - \frac{R \cdot M}{N} = \frac{r \cdot m}{n}\left\{1 - \frac{(1+c)(1+d)}{(1+b)}\right\},$$

and as $\frac{r \cdot m}{n}$ is the number of accidents per employee in 1906,

$$\left\{1 - \frac{(1+c)(1+d)}{(1+b)}\right\},$$

is the rate of decrease. In the problem given, $(1+b) = 1.66$ and $(1+c)$ is 1.25, while the value of the whole expression is .46. Then

$$1 - \frac{1.25 \times (1+d)}{1.66} = .46,$$

or

$$1 + d = \frac{.54 \times 1.66}{2.25} = .71712.$$

The rate of decrease in the number of accidents per machine is .28288, or about 28 per cent. We are told that during the five years there were no fatal accidents and no loss of limbs or eyes, and it seems reasonable to suppose that the rate of accident per machine, showing a decrease of 28 per cent., is a fairer indication of the improvement in safeguarding machines than the decrease in accidents per one thousand employees, which amounted to 46 per cent.

Mr. Mowbray's idea that improvements in the safeguarding of machinery and the lessening of the risk of accident will most surely follow the adoption of premium rates based upon the degree of perfection in the safeguards employed, is undoubtedly true. It is similar to the effect of the system of mercantile schedule rating in use by fire insurance companies. The real question raised by this paper is whether the data obtained from the experience in a country like Germany may be employed in America, where the protective principle has not been so fully developed. This seems to me to be analogous to using a mortality table with interest at three per cent. and at an age ten years in advance of the true age, in order to obtain a three and a half per cent. net premium and allow for an excessive mortality. The use of such a table is not as satisfactory as employing a table showing the true mortality rates and the correct rate of interest, but lacking such a table the use of the approximate measuring stick is allowable.

Mr. Mowbray recognizes that a great deal of responsibility must be placed upon the safety engineers in determining the rates to be charged in practice. If the method suggested by Mr. Mowbray enables these engineers to exercise their discretion in determining the various elements of the rates, instead of making a rough guess at the rates as a whole, the German statistics may prove of value until we have our own data to guide us. The Society is indebted to Mr. Mowbray for bringing this question to our attention.

MR. MOWBRAY:

(AUTHOR'S REVIEW OF DISCUSSIONS.)

Mr. Papps has pointed out that on the figures quoted as regards prevention work in the Remington Typewriter works a rate of decrease in the number of accidents per machine of 28 per cent. is a truer index of the improvement in safeguarding machines than the decrease in accidents per 1,000 employees, which amounted to 46 per cent. In this Mr. Papps is correct, but according to the summary of German experience for the year 1907, as published in Bulletin 92 of the Bureau of Labor, in the group of associations for insuring manufacturers of "fine mechanical products," the per cent. of persons killed and injured, due to working machinery, was 37.00 for the year 1907 and 39.15 for the year 1897. While working machinery is the most important single cause of accidental injury, the improvement in working conditions, discipline and general safety regulations is probably better represented by the decrease in accidents per 1,000 employees.

Mr. Forbes points out the difficulties at the present time in the way of actually applying any system of merit rating to compensation risks. He further points out that under present conditions a system of rates, under which reductions are made for favorable conditions, is a more workable system than one in which charges are made for departures from the ideal risk. Undoubtedly, as long as the competition between insurers is as keen as it is at the present time and the desire to make concessions in order to obtain the business of a given employer rather than have it go elsewhere is strong, this condition will apply.

He further points out that "liability underwriting in the past has not been, as often claimed, entirely a matter of guesswork," and that the liability companies have available a large fund of information tending to show the comparative hazard of various American industries. The problem of the proper rate for compensation insurance is not one which concerns the liability underwriter alone, and this data is available to him only, for the liability companies have not seen fit to place this fund of information at the disposal of the general public. Whether in this they have been wise or not, it remains for the future to determine. In the absence of such information, social workers and others have turned to the European statistics as a basis for their estimates. If, as is claimed, and is probably true, European risks, and particularly German risks, are better risks than the American, then European statistics can only be used for determining the rates upon an ideal or standard risk which will closely correspond to the European. For the conditions in which the American risk falls short of this standard, the rates so brought out must be increased according to a reasonable estimate of the greater probability of injury and loss. Mr. Forbes

notes that the rates in the Illinois Manual for wood-working establishments vary from $5.60 to $1.85, as against a net rate of 79½ cents, plus the necessary loading for medical aid and expense, deduced by me from the German data. When it is remembered that the cost of medical attention represents approximately 40 per cent. of the compensation cost and that the manual rates were computed upon the assumption of a 60 per cent. loss ratio, it will be seen that the net rate of 79½ cents is equivalent to a manual rate of about $2.00, and this as an average rate for the highly protected plants as against the manual rates for the average plant. The difference in estimates, therefore, is not so marked as appears at first glance. As was pointed out on page 225, this *average* rate was worked out for the wood-working industry because the summary of the data did not further divide this industry, although, if the plan suggested were to be adopted, further subdivided data should be obtained.

Mr. Forbes expresses the view that, if American statistics were used, the ratios presented in Table 4 would be considerably changed. In this I agree, for the ratios in that table are determined upon the basis of German establishments in which accident prevention work has been going on for upward of twenty years. It seems altogether likely that the possible effect of this work is more limited as regards certain causes than others. Those causes of injury whose force can be most reduced by prevention work would probably be much more effective in the average American risk to-day than in the ideal upon which the basic rate is based.

To those who have watched the development of the competition between mutual life companies and those issuing non-participating business, Mr. Forbes's closing remarks as to the probable tendency in regard to compensation insurance are particularly interesting.

In closing it is perhaps as well to again point out that at least one purpose of presenting this suggestion was to point out to those who are without practical experience in insurance work, yet have occasion to investigate these problems, what seems to be the one safe method of procedure in applying European data to our American problems.

THE MORTALITY EXPERIENCE OF THE MUTUAL BENEFIT LIFE INSURANCE COMPANY ON PAID-UP POLICIES ISSUED IN LIEU OF SURRENDER POLICIES—E. E. RHODES.

VOL. XIII, PAGE 233.

WRITTEN DISCUSSION.

MR. GORE:

In presenting to the Society the mortality experience of his company under paid-up policies in such a way as to make possible an easy comparison with the extended insurance experience of the same company presented in 1908, Mr. Rhodes has rendered a real service.

There is a very small difference between the average exposures of policies issued since 1879 and those issued earlier. This leads to the supposition, confirmed by examination of the various tables presented by Mr. Rhodes, that the great bulk of the paid-up issues since 1879 were in the years immediately following that date, notwithstanding the very large increase of the business in force in subsequent years.

It follows that the custom of asking for paid-up policies must have been much more common in the eighties and previously than it is now. Probably the explanation is that the availability of cash surrender values was not so well known, and possibly their amounts were less attractive than at present. It would seem, therefore, that a large number of those who took these paid-up policies correspond to the class who now take cash surrender values.

This leads to the conclusion that while the experience is not one of paid-up policies issued under modern conditions, it does afford some indication of that much sought desideratum—the rate of mortality among surrendering policyholders.

From this point of view Table D affords some very interesting information. As the calculations of the expected mortality by the three select tables were based on duration from original entry and not from date of becoming paid-up—so that the expected mortality in respect to a given age at issue and a given duration of a paid-up policy was not constant—the figures relating to the American Table will be of most value.

The selection against the company is very clearly marked, the first year of duration showing ratios of the American expectation of only 51 per cent. for policies, and 62 per cent. for amounts, in the total experience, and 48 per cent. and 69 per cent., respectively, in the experience of issues since 1879. Remembering that this

first policy year's experience is mainly that of a quarter of a century or more ago, when the general mortality was higher than to-day, there seems little reason to doubt that these individuals were quite equal to newly selected lives of the same ages.

A very curious feature is that the mortality by amounts is heavier than by policies. This feature is much more pronounced in the business prior to 1880 than in the subsequent issues, but it is so persistent throughout that its appearance can hardly be accounted for as an accident.

Variation in amount among paid-up policies may arise from variation either in the amount or in the duration of the orignal insurance. Results according to duration prior to becoming paid-up are set forth in Table B, and this table, for all issues, shows a slight tendency to rising mortality percentages with increase of duration prior to becoming paid-up, but does not do so for issues since 1879. Separating the issues prior to 1880 from the total, we obtain the following summary:

Paid-up Policies Issued Prior to 1880.

Duration of Original Insurance.	Percentage of Actual to Expected Deaths by Policies.
2–5 years	75.9
6–10 years	81.2
11–15 years	81.7
16–20 years	89.1
21–44 years	100.2

The above table indicates that, in the earlier issues, policies larger than the average, by reason of longer duration of the original insurance, were subject to a higher percentage of the American mortality than the smaller policies issued after shorter durations of the original policies. All automatic paid-up policies having been excluded from the experience of the earlier issues, it is probable that some of the deaths under the smaller policies have not been reported.

The paid-up policies derived from larger insurances have shown a somewhat higher percentage of the American mortality than where the original policies were for smaller amounts. This may be explained, in part at least, by the fact that the normal mortality at the younger ages is a considerably smaller percentage of the American expectation than at the older ages, and that the proportion of sums insured issued at older ages is greater than the proportion of policies, because the average policy increases with increasing age. As an instance of this increase, a random group of 1,058 whole life insurances gave the following average policies:

Age of Entry.	Average Policy.
15–19	$1,000
20–24	1,698
25–29	1,851
30–34	2,045
35–39	2,095
40–44	2,709
45–49	3,188
50–54	3,853
55–59	3,488
60–64	1,416

A simple illustration of the effect of differences, such as those shown in the above table, may not be out of place. Suppose 1,000 policies averaging $1,500 each, issued at age 20, and a further 1,000 averaging $3,000, each at age 50; the expected mortality by the American table is:

POLICIES.

$1{,}000 \times q_{20} = 7.8$
$1{,}000 \times q_{50} = 13.8$
Expected deaths, 21.6

AMOUNTS.

$\$1{,}500{,}000 \times q_{20} = 11{,}708$
$3{,}000{,}000 \times q_{50} = 41{,}343$
Expected claims, 53,051

Now, if the actual mortality at age 20 is 40 per cent., and at age 50, 60 per cent. of the American expectation, the results will be as follows:

POLICIES.		AMOUNTS.	
Age 20	3.12	Age 20	4,683
Age 50	8.28	Age 50	24,806
Actual deaths	11.40	Actual claims	29,489

The ratio of actual to expected is 52.8 per cent. by policies, and 55.6 per cent. by amounts, although large and small policies have at each age been subject to identical mortality rates.

MR. HUTCHESON:

We are very much indebted to Mr. Rhodes for having prepared this paper for The Actuarial Society, and for having given us the ratios of the actual to the expected deaths for policies and for amounts insured by the four different standards.

The object of the investigation was, apparently, in the first place, to determine whether or not there had been any selection by the insured against the company in taking paid-up insurance, and, in the second place, to determine how the experience under the paid-up policies compared with the experience under the extended insurances.

Mr. Rhodes explains that he excluded the limited payment policies, which by their terms became automatically paid-up for proportional parts of the original amounts insured. It is not clear to me why these policies were excluded.

Was there any Selection against the Company in taking Paid-up Insurance? Regarding the first object of the investigation, namely, to ascertain whether or not there had been any selection against the company in taking out paid-up insurance, the figures which seem to bear most directly on this point are those contained in the tables on page 244 of Volume XIII for the paid-ups and those on page 610 of Volume X for the general experience of the company. Table B^1, on page 244, gives the percentages of the actual to the expected mortality by the four standards for paid-up policies where the original policies had been in force over one year, over two years, over three years, etc., before the paid-up was granted. Table B^1, on page 610 of Volume X, gives us a table of similar percentages of actual to expected by the American Table for the company's general experience, commencing, however, with policies which had been over two years in force. This latter table is for amounts insured.

The following table gives the percentages in parallel columns for the paid-ups and for the general experience:

EXTRACTS FROM TABLE B.1

After Being in Force Over	Paid-ups Issued Since 1879.	Total Paid-ups.	General Experience.
2 years	85.58	87.49	89.7
3 "	85.31	87.29	90.9
4 "	85.23	87.09	91.9
5 "	85.64	87.35	92.8
6 "	85.30	96.93	93.7
7 "	85.30	86.23	94.5
8 "	85.21	86.11	95.2
9 "	86.04	86.49	96.0
10 "	86.14	86.17	96.8
11 "	86.80	86.63	97.2
12 "	86.46	87.50	97.7
13 "	86.37	87.98	98.0
14 "	88.21	89.91	98.2
15 "	87.67	89.28	98.5
16 "	88.70	89.97	98.7
21 "	89.13	90.15	99.4
26 "	84.93	81.61	98.3

It would appear from this table that the experience under the paid-ups was considerably more favorable than the general experience of the company.

I have taken the figures in Tables B^1, as these are the only tables (page 610, Volume X) which give the percentages of actual to expected for the company's general experience, and for paid-ups on the same basis.

I think that the comparison is probably a fair one, as the paid-up policies would extend more or less over the same policy years as the general experience of the company.

Comparison of Mortality Experience under Paid-up Insurance and Extended Insurance.—Mr. Rhodes gives a table, on page 236, comparing the experience under extended insurance with that under paid-up insurance, grouped according to the number of years the original policies had been in existence before the extended insurance or the paid-up insurance went into force. In the first group, we find the policies on which one to five premiums had been paid; in the second, those on which six to ten premiums had been paid; in the third, those on which eleven to fifteen had been paid; and so on. Those on which one to five years' premiums had been paid before the extended insurance was granted would be entitled to extended insurance for a comparatively short period of years, whereas the paid-up insurance would run for the remainder of life. The extended insurance would, therefore, expire a comparatively few years after the medical examination for the original policy, and the paid-up insurance, on the other hand, would continue until the limit of life. It does not seem proper, therefore, to make the comparison shown in this table on page 236. It appears to me that the comparison should be made policy-year by policy-year.

The following extracts from Tables D show the difference in the incidence of the insurance under the extended insurance and under the paid-up insurance; they are taken, respectively, from pages 613 of Volume X and 246 of Volume XIII:

FIRST EXTRACTS FROM TABLE D.

Year of Exposure	Extended Insurance.	Total Paid-ups.
1	$61,993,929	$7,524,055
2	44,889,622	7,105,854
3	32,638,691	6,661,867
4	24,083,789	6,233,709
5	17,602,878	5,882,948
10	3,462,400	4,259,105
15	265,300	2,877,911
20	2,000	1,803,575

Tables D, on pages 246 and 247, give us the percentages of the actual to the expected mortality for each policy-year during the existence of the paid-up policies, and Table D, on page 613 of

Volume X, gives us the corresponding percentages for the extended insurances. The following table gives these percentages for each policy-year according to the Compound Progressive Table for amounts insured for (1) total paid-up policies, (2) paid-up policies issued since 1879 and (3) extended insurances:

SECOND EXTRACTS FROM TABLE D.

Year of Exposure.	(1) Total Paid-ups.	(2) Paid-ups Issued Since 1879.	(3) Extended Insurance.
1	64.31	70.52	124.8
2	88.62	80.98	102.5
3	83.76	78.14	85.7
4	81.99	94.28	90.3
5	76.70	78.64	112.2
6	104.53	95.43	88.2
7	83.86	85.15	96.0
8	81.56	80.76	88.0
9	89.05	97.08	92.2
10	91.56	98.79	98.5
11	81.19	82.67	126.5
12	84.90	72.66	84.9
13	109.06	122.63	154.0
14	76.18	57.34	47.0
15	110.59	95.18	142.1
16	88.26	95.06	524.7
17	91.18	79.38	
18	114.36	113.79	
19	95.82	93.66	
20	107.05	80.64	
All years	87.89	85.95	103.3

The above extracts from Tables D show that the amount insured in the first year under extended insurance was about $62,000,000 as against $7,500,000 of paid-up insurance, and that the mortality rates were 124.8 per cent. and 64.31 per cent., respectively. As the amount of extended insurance rapidly diminishes it is evident that the early years of exposure are the important years.

A comparison of the amounts insured under the extended insurance and under the paid-up insurance, and a glance at the percentage of actual to expected deaths, will clearly show that the paid-up insurance was much less expensive to the company than the extended insurance.

Commenting on the table given on page 236, Mr. Rhodes states in the last paragraph on page 237, that, " When the comparison is made with reference to the years between the date of issue of the original policy and the issue of the paid-up policy or granting of the extension, as the case may be, it will be seen that the general tendency is for the experience under paid-up policies to be in excess of that under extended insurances, where the paid-up or the exten-

sion occurs within a few years of the original date of issue." This seems to me an erroneous conclusion to draw from the figures given on page 236, for the reasons above stated.

In the first paragraph on page 241, Mr. Rhodes states that there appears to be a selection against the company in the case of extended insurance with a heavy mortality in the first few years, whereas, in the case of the paid-up insurance, the mortality is not felt for some years. Further, he states that the mortality under the paid-up policies is nearly as high as under the extended insurances; the table on page 240 does not seem to confirm this conclusion.

The conclusions which I reach from reading Mr. Rhodes's paper are, in the first place, that those taking paid-up insurance did exercise a selection against the company by taking paid-up insurance rather than continuing their original policies, for the mortality under the paid-up insurance was less than under the company's general experience; second, that the experience under the paid-up insurance compared very favorably with that under the extended insurance; and finally, that paid-up insurance is much less expensive for a company than extended insurance.

MR. RHODES:

(AUTHOR'S REVIEW OF DISCUSSIONS.)

Mr. Hutcheson refers to the fact that policies which by their terms became automatically paid-up for proportional parts, were excluded from the investigation and he asks whether there was any selection against the company in taking paid-up insurance. The paid-up policies issued automatically were excluded for the reason that in most cases the insured simply neglected to pay premiums, and did not exercise any option as was the case under all other paid-up policies issued by the company.

Mr. Hutcheson objects to the comparison shown on page 236 for the reason that the duration of extended insurances is less than that of paid-up policies. This objection holds good in so far as the first two lines of the table are concerned, but after a policy has been more than ten years in force, the effect of medical selection is generally supposed to have been practically eliminated, so that for those policies extended or surrendered after being more than ten years in force the comparison seems proper.

The comparison shown on page 240 gives the result of the first twenty years' exposure. In the summary the years of duration should be "1–20" instead of "1–end." This explains the slight difference between the figures given on page 240 and those shown by Mr. Hutcheson in the comparison he gives. A comparison by the Compound Progressive Table differs considerably from a comparison by the Modified English. It is interesting to notice the difference for all durations.

PERCENTAGE ACTUAL TO EXPECTED MORTALITY.

	Modified English.		$O^{[M]}$.		Compound Progressive.	
	Policies.	Amount.	Policies.	Amount.	Policies.	Amount.
Extended.....	92.0	97.5	88.9	92.7	99.6	103.3
Paid-up after 1879.....	92.69	97.35	82.73	86.69	84.18	85.95
Paid-up Total ..	91.09	98.98	81.05	87.99	82.26	87.89

This comparison shows that in comparing the mortality of two companies, or different groups of business of the same company, the results will sometimes depend very largely upon our choice of a standard table. I agree with Mr. Hutcheson that owing to the different distribution of the exposures under extended insurances and paid-up policies, the former will be more costly to the company. I am not inclined to agree with his conclusions, that the mortality under paid-up policies is more favorable than the general experience of the company. I do not believe that we have the necessary data before us to enable us to draw any such conclusion with safety.

THE BASIS FOR EMPLOYERS' CONTRIBUTIONS TOWARD SERVICE PENSIONS—A. H. MOWBRAY.

VOL. XIII, PAGE 248.

WRITTEN DISCUSSION.

MR. EDWARD B. FACKLER:

Mr. Mowbray has outlined an ingenious plan for replacing worn-out employees by new men of full efficiency, and paying pensions to those retired, without increasing the outlay by the employer. It is to be regretted, however, that in his illustration of the practical working of his system, he did not indicate its results where the efficiency of the retiring employee is higher, and closer to that of the man who takes his place. His illustration makes the efficiency of the former 45 per cent. and of the latter 95 per cent., and in such case the enforced retirement at 65 would carry a life pension of about 39 per cent. of the average salary, which otherwise would have been paid up to age 75. This plan would be useful in dealing with actual conditions when a pension system is just being adopted; it would also serve where accident or incapacitating illness causes a sudden sharp loss of efficiency. Where, however, the loss of efficiency is gradual, the employer might refuse to wait until the 45 per cent. degree of efficiency is reached, for additional outlay would be necessary to get the same amount of work done. If earlier retirement were enforced under such circumstances, the resulting pension might cause hardship.

Any plan such as this suggested by Mr. Mowbray, involves the theory that a long-time employee has a moral right, during good behavior, to full salary or its equivalent until, say, age 75, in order that he may save a competence for old age. If this be granted, it may be claimed with equal propriety that the employers are entitled to see that a part, at least, of the yearly wage shall be definitely applied to that end. And this can be done without interfering with the operation of Mr. Mowbray's plan. The writer has had occasion to make some such calculations, which have been put into actual use.

Using Mr. Mowbray's conventional retirement age, 75, as the objective, a small percentage of each year's salary, beginning about age 60, may be given to the employee, not in cash, but in a contract for a deferred life annuity with monthly payments to begin at age 75. Each contract would become definitely the property of the employee and would not be forfeited for any cause. Then if the

employee's failing efficiency were to render his earlier retirement necessary under Mr. Mowbray's plan, the future contributions by the employer to pensions might be made less than otherwise.

Assuming that the employers maintain their own annuity fund, with American 4 per cent. as the basis of calculations, and that $5 a month, or roughly $60 a year, is apportioned to purchase annuities, the following results would be obtained, by yearly purchases:

MONTHLY ANNUITY BEGINNING AT AGE 75 PURCHASED BY $60 APPLIED AS A SINGLE PREMIUM.

Age at Purchase.	Monthly, Beginning at Age 75.	Age at Purchase.	Monthly, Beginning at Age 75.
60	$3.73	68	$2.03
61	3.49	69	1.85
62	3.26	70	1.68
63	3.03	71	1.51
64	2.82	72	1.36
65	2.61	73	1.21
66	2.41	74	1.07
67	2.21		
Total 8 annuities	$23.56	Total 15 annuities	$34.27

Judging by existing rate tables, the annuities which regular insurance companies would offer would fall far short of those given above, but it seems that contracts to cover all employees of an establishment might be made on a satisfactory basis. The element of selection against the company would be inconsiderable, for all of a certain attained age, whose efficiency had not fallen below a certain minimum, would be covered without further regard to physical condition. Such a plan would open a new field for insurance companies in serving the needs of the public.

MR. FLYNN:

Mr. Mowbray has directed our attention to a most interesting phase of the general problem of staff pension funds. The point which Mr. Mowbray brings out in a clear and convincing manner is, briefly, that there is a monetary gain to the employer who installs a plan for retiring old inefficient employees on part salary, and, further, that this gain is a reasonable basis upon which to measure the employer's contribution to the cost of such pensions. It is probable that every one will admit the truth of the first part of this statement, and will agree that the monetary gain of the employer is a proper standard for the measurement of the amount of his contribution to the cost of the plan. The question immediately arises, however,—"Can we obtain a sufficiently accurate measure of the efficiency of a staff to use as a basis for the calculation of the gain?"

Before discussing the problem from this standpoint, I would like to dwell for a moment on one feature of the calculation. Mr. Mowbray has used as an illustration the case of an employee aged 65, who is replaced by a man of younger age whose efficiency rating is higher by fifty points, and from this case has established the mathematical basis of the calculation. In this example it is assumed that a fixed efficiency rating can be given to a man for a future term of years; for, in referring to formula (1), e_x is defined as "the probable efficiency rating for the next $(y-x)$ years of the man to be retired." Now, the efficiency of an employee diminishes rapidly year by year after a certain age has been attained—the age being lower for physical workers than for those whose work is principally mental —so that the value of the factor $(e'-e_x)$ in formula (1) will increase rapidly as x increases beyond 70 or 75. I would point out, therefore, that, although for purposes of illustration a probable efficiency rating for a future term of years may be assumed, in the practical working of the problem it will be safer not to attempt to fix this efficiency rating for a term of years, but to determine the percentage of loss through inefficiency at each age and to discount this loss according to a mortality table and interest.

This method of handling the problem is well illustrated in the work of the President's Commission on Economy and Efficiency in its attempt to ascertain the annual loss to the government through inefficiency of aged employees.*

In this investigation, the Commission found that, upon the basis of returns made by department heads, the average inefficiency of clerks in the government service ranged from 14.26 per cent. at age 70 to 51.30 per cent. at age 85 and over. After being graduated by a modification of Woolhouse's formula and then by the graphic method, the percentages of inefficiency at quinquennial ages, beginning at age 70, were as follows:

Age.	Per Cent.
70	14.4
75	19.6
80	26.0
85	35.2
90	57.5
95	100.0

The annual loss to the government through inefficiency was then obtained by discounting upon the basis of a mortality table the amount of salary which was not earned by each old employee.

As Mr. Mowbray states, the result of the attempt of the Commission to ascertain the annual loss to the government through ineffi-

* See "Retirement from the Classified Civil Service of Superannuated Employees." Government Printing Office, 1912.

ciency of superannuated employees was unsuccessful. The reason for this result lay in the fact that although plans were laid so that the returns of the heads of departments and bureaus would be entirely impersonal, nevertheless, there was reluctance upon the part of the officials to make returns in particular cases of gross inefficiency. It was estimated by the Commission that the returns "did not show much more than half the actual loss sustained by the government."

The lesson to be drawn from this investigation of the Commission would seem to be that it will be unsafe to rely upon personal estimates of the efficiency of employees and that it will be necessary to fix upon standards by which the efficiency of employees, both as to quantity and quality of work performed, can be measured.

An excellent example of a successful investigation of this kind is given in the report of the President's Commission on Economy and Efficiency on "Methods of Keeping Efficiency Records of Employees in the National Bank Redemption Agency of the Department of the Treasury."

The work of the employees of the National Bank Redemption Agency is to sort and count paper money which has been sent in for redemption. It is a line of work which is particularly well adapted for ascertaining a reliable standard of efficiency and for using it to measure with mathematical accuracy the efficiency of each clerk, both as to quantity and quality of work.

The report of the Commission is an interesting document which shows clearly what can be accomplished in the field of efficiency rating in certain lines of work. Because of the success of this investigation the government has undertaken, under the direction of Mr. Herbert D. Brown, who had personal charge of the earlier investigations of the Commission along this line, to work out a system of efficiency rating throughout the government service. It will be interesting to note the methods used and the degree of success which will be attained by the government in handling this difficult problem.

The report of the Commission, just referred to, presents a plan by which each employee will be paid what he earns as nearly as the salaries appropriated by Congress will permit. It is worthy of note, as bearing upon the idea presented by Mr. Mowbray, that if a plan could be worked out by which each employee would be paid what he earned, there would be no monetary gain to the employer in installing a retirement plan.

It is probably safe to assume, however, that in view of the weight of public opinion on the relations of employer and employee in this country, the establishment of reliable standards for measuring the inefficiency of workers will not result in the modification of the pay of those workers who have become inefficient through old age.

MR. MOWBRAY:

(AUTHOR'S REVIEW OF DISCUSSIONS.)

Mr. Fackler points out that in the illustration of the practical working of this suggestion a narrower margin of efficiency rating between the retiring employee and the man who takes his place would be preferable. In this I agree with him, for, among other things, it would have kept more clearly in mind the fact that the suggestion was only intended to apply to the cases of a pension fund maintained by the contributions of both employer and employee, a fact which the objection brought out against the illustration tended to obscure. While, as Mr. Fackler states, the loss in efficiency is gradual, the gain is abrupt in the beginning, even though, as Mr. Flynn notes, it is not constant for the interval between the actual retirement age and the assumed retirement age forming the basis of compilation.

Mr. Fackler's suggested use of an annuity as a means of dealing with the employees' contribution is interesting, but unless begun late in life presents some of the difficulties encountered in other plans when the question of withdrawal from service is under consideration. From this point of view the so-called savings bank plan has much to commend itself.

Mr. Flynn is right in pointing out that the value of the factor $(e'-e_x)$ tends to increase rapidly as x increases beyond seventy and seventy-five. In presenting the suggestion it seemed better not to complicate the matter by undertaking refinements of this kind, which would promptly present themselves when a particular problem was under consideration.

I quite agree with Mr. Flynn that the fundamental requirement for the application of the suggestion presented is a sufficiently accurate measure of the efficiency of the staff for use as a basis for the calculation of the employers' gain through the retirement of inefficient members, and in his further deduction from the efforts of the President's Commission on Economy and Efficiency to measure the loss to the government "that it will be unsafe to rely upon personal estimates of the efficiency of employees and that it will be necessary to fix upon a standard by which the efficiency of employees, both as to quantity and quality of work performed, can be measured."

Mr. Flynn is also right in pointing out "that if a plan could be worked out by which each employee would be paid what he earned, there would be no monetary gain to the employer in installing a retirement plan," but, as he points out, there are other conditions which will probably prevent the application of this principle to those workers who become inefficient through old age.

SELECT AND ULTIMATE MORTALITY GAIN ON SINGLE PREMIUM POLICIES—EDWARD W. MARSHALL.

VOL. XIII, PAGE 256.

WRITTEN DISCUSSION.

MR. THOMPSON:

The methods here set forth by Mr. Marshall, coupled with those proposed by Mr. Henderson in The American Underwriter and by Mr. Sheppard in Volume IX of the *Transactions of the Actuarial Society,* complete a system which enables one to determine the mortality margin on almost any form of policy in an extremely facile manner, and there is little to be said by way of comment thereon. It may be noted that the generality of Mr. Marshall's formula extends to policies on two or more joint lives, in which case the mortality margin becomes, in case of two lives of equal age,

$$\frac{M_{xx} - M_{[x][x]}}{D_{[x][x]}} - A_{xx\overline{n}|}\left(\frac{l_x^2}{l_{[x]}^2} - 1\right),$$

or, if the select and ultimate M column be not available,

$$(1 - A_{xx\overline{n}|})\left(\frac{l_x^2}{l_{[x]}^2} - 1\right) - d \cdot \frac{N_{xx} - N_{[x][x]}}{D_{[x][x]}},$$

(in case the function $d\frac{N_{xx} - N_{[x][x]}}{D_{[x][x]}}$ should be tabulated) although the limitations on the use of this form of policy make the observation of scarcely any practical force.

The advantage of uniformity in the calculation of mortality margins is at once admitted. It may further be of interest to compare the number of operations required by Mr. Marshall's method with those required by the usual more direct formula for calculating these mortality margins on various odd forms of policies. Consider together the 10 payment life and the single premium 10 year endowment forms. The "primitive" method requires the calculation of temporary annuities and involves a subtraction and a multiplication at every age, each to be done in duplicate, four operations in all.

The select single premium endowment net premiums can be calculated therefrom by the formul $1 - d\mathrm{a}_{[x]:\overline{10}|}$ which admits the application of a continuous method, so that it may be said that the select single premiums are derived in a single additional operation. The calculation and the checking of $A_{x:\overline{n}|} - A_{[x]:\overline{n}|}$ add two more operations, making seven in all. The computation of the mortality margins for 10-year term policies by the old formula requires eight operations, including checking. The use of Mr. Marshall's method on single premium endowments needs only four operations, and Mr. Henderson's method applied to a 10-year term policy likewise can be carried out and checked in four operations. On the other hand, there is to be considered the preparation of tables of values of the functions

$$\frac{N_x - N_{[x]}}{D_{[x]}},$$

$$\frac{M_x - M_{[x]}}{D_{[x]}},$$

$$\frac{l_x}{l_{[x]}} - 1.$$

The first two involve a subtraction and a division and the last a division only (ignoring the subtraction of unity as an operation), a total of ten processes, including duplicate operations for checking. Thus the old method requires fifteen operations and the proposed new method eighteen for the derivation of mortality margins on the two forms mentioned. However, the advantages of the new method are quickly seen when the permanently established functions are used in further calculations, so that if it be found necessary to calculate the required values on two or more "pairs" of forms related as above indicated, the reductions in work covered by the newer and more elegant methods of Messrs. Marshall, Henderson and Sheppard are worthy of note. It may be observed that the twenty-five and thirty payment life policies may be "paired" with the single premium 25 and 30 year endowment policies in respect to the requirements for the calculation of 25 and 30 payment temporary annuities, the functions ${}_{25}G_x$ and ${}_{30}G_x$ not being published.

MR. E. W. MARSHALL:

(AUTHOR'S REVIEW OF DISCUSSION.)

Mr. Thompson's remarks do not require an extended reply, but one or two points may be touched upon. In finding the mortality gain on joint policies, it is not unusual to approximate to the same by reducing the ages to an equivalent single age, by Simpson's rule,

and to treat the policy as based on that single age—a very simple method in practice.

The numbers of operations required in applying the various formulas mentioned may be reduced by calculating the general expression P.Q. — R.S. on the arithmometer, throwing the product P.Q. upon the movable slide, and subtracting the product R.S. therefrom without clearing the machine.

MODERN SURRENDER VALUES—JAMES F. LITTLE.

VOL. XIII, PAGE 259.

WRITTEN DISCUSSION.

MR. MOIR:

Probably the central point of this paper is the discussion of the question as to whether or not policyholders who surrender their contracts are healthier or less healthy than the average. This question should have much weight in fixing the basis of surrender values; it is one of the important actuarial questions, and one to which I have given attention for many years. Mr. Little's argument in regard to Mr. Chatham's discussion on the difference in mortality between the Scotch and the British offices is generally sound. Like Mr. Little I feel that those figures do not support the argument of Mr. Chatham; but I would not go quite so far as Mr. Little in claiming that they prove the opposite—in my judgment they prove nothing. On the other hand, the figures submitted at an earlier date—1870—by Dr. Sprague were shown to be unsound by Mr. Macaulay (*J. I. A.*, XXXII, 117); indeed this is one of the few instances in which Dr. Sprague seems to me to have used superficial rather than sound and profound reasoning. His work was generally so perfect, both from the practical and scientific standpoint, that this particular lapse is all the more noticeable. We come back to Mr. Little's statement that we have only circumstantial evidence on which to base an argument. In other words, the question is one of opinion in the weighing of probabilities which cannot be determined scientifically.

It depends entirely on Mr. Little's premises as to whether or not I agree with his conclusions. If his arguments are premised upon conditions as they now exist and as they have been in the past (and this is what I should suppose from the tone of his remarks), then my opinion is the contrary of his. But if his opinion is based upon present-day practice in America in the matter of surrender values, and in prophetic utterance as to the probable effect of such full surrender values in future, then I agree with his conclusions.

My reason for these two apparently opposite views can only be given briefly. I had long experience with the actual work connected with the surrender of policies. I found that the reason given for surrender was in nearly every instance: "I need the money." Mr. A. D. Besant investigated this point, and out of 1,085 cases of surrender found that 840, or 77.4 per cent., "were given up entirely

from impecuniosity" (*J. I. A.*, XXXV, 234). On more occasions than one I have paid a surrender value to a man who was in such a condition of semi-intoxication that he could little more than write his name, in many cases to policyholders with an appearance of dissipation. My first argument, therefore, against Mr. Little's view is personal observation. The second argument is along similar lines but deals with the general question of human nature. A thrifty man knows that under conditions as they have been in the past he loses by surrendering a policy. The class, therefore, maintaining insurance most tenaciously is the thrifty, which is also a long-lived class. Mr. Little says that, "wherever the insuring public have an opportunity of exercising selection favorable to themselves, etc." With this statement I agree, but as conditions have been in the past, the surrender of a policy meant exercising an option not favorable but *unfavorable* to the thrifty policyholder, and the thrifty policyholder knew it. The third and last point relates to the observation of others: I have discussed the subject with many of the oldest and clearest thinkers in life insurance, and have found in the past twenty years a strong tide of feeling towards the view that those who surrender are not such good lives as the average—Dr. Sprague does not nervously recede from a position which he can consistently hold, yet in 1895 he said: "I now entertain considerable doubt on this subject" (*J. I. A.*, XXXII, 199).

But for the future the situation will, in my judgment, be very different. Most companies guarantee the full reserve value after the lapse of a period of years—some of them after three years only. It follows that a policyholder may then surrender such a policy *without personal loss,* and transfer to another company, sometimes indeed to his positive advantage. In future, therefore, to again use Mr. Little's words, "the insuring public will have an opportunity of exercising selection favorable to themselves." In some cases such selection may be exercised when it may not be favorable to themselves, but in all cases it will at least be favorable to the agent who prevails on them to make the change. I fear greatly that the payment of full reserve values under policies will bring forth and nourish a class of twisting agents, and bring about the condition that the better lives will frequently surrender and transfer their policies. Some agents will be actuated by honorable motives, doing what they consider their duty in transferring a policy from a weak to a strong company; others will follow the profession for what they can make out of it, whether they act in the interests of their clients or not. Ten years hence we shall find that there are many policies which could be changed with advantage to the holder, and then we may look for adverse selection. The opportunity is wide open since agents will make a profit out of the deal.

It may be that Mr. Little and I attain our results from a very different basis of argument, but apparently our conclusions are nearly the same. Like him I deprecate the payment of the full reserve value

on surrender, except for paid-up insurance. There is, however, one way whereby a company may protect itself and still pay the full legal reserve; it is by very strict conservatism in the distribution of surplus, and the carrying forward of a good contingency reserve. While the policyholder can withdraw the entire *legal* reserve, he would not obtain any share of the *contingency* reserve, and, therefore, a company can take precautions against being seriously weakened. If the entire reserve can be withdrawn, then the effect of transfers of policies to stronger companies would further weaken those which are below the best. Such companies will have a constant struggle to maintain by "new blood" the vitality they lose by such transfers; and, while even the best will suffer, we shall see yet another illustration of the ancient adage: "To them that hath shall be given, etc."

My position on this question of surrender values has been reversed in recent years. From ten to twenty years ago I was a strong advocate of more liberal values; now that the values have moved up radically in this country I find myself advocating smaller values. *Briefly then, my view now is that while adverse selection on surrender has not existed in the past, yet, to avoid its effects in future, some surrender charge, or, alternatively, a large contingency reserve, is most desirable and the only course of wisdom.*

With reference to the retrospective method of calculating values for surrender, I find that my views, as expressed many years ago, have not changed and therefore quote as follows:

> "Much may be said in favor of the retrospective method being applied to the calculation of surrender values in the early years. It admits of a good defence theoretically, facilitates the making of a proper allowance for initial expenses, and returns to the policyholder his effective contributions accumulated at the rate of interest assumed by the office. The formulas may not, however, be so simply applied as those of the prospective method, and this would prove a serious objection, especially when it is borne in mind that the latter method may be adjusted so as to give somewhat similar results. One great merit of the prospective method and the use of net values is that the system applies throughout the entire duration of the policy. If the retrospective method and office premiums less actual expenses be employed, then a slight modification in the interest basis completely alters the accumulated amount when many years have elapsed, but this objection is of little importance in the early years" (*J. I. A.*, XXXVI, 32).

There are many incidental statements made by Mr. Little to which I find I cannnot subscribe, but probably a general indication of this is sufficient without entering into trivial details. For example: on page 269 Mr. Little refers to "the small section of policyholders in decidedly bad health . . . none of whom, practically,

ever surrender." Close observation and association with the surrender department of an insurance company has proved to me conclusively that this is not so. Some men in very bad health surrender their policies. Again, on page 271: "The contingency reserve has a different function, viz.: that of providing for a possible depreciation in assets." I cannot see any good reason for so limiting the contingency reserve which may be applied to many other objects. These are two of the more important remarks selected from many minor points on which I find that I am not in harmony with Mr. Little's views.

MR. WENDELL M. STRONG:

One of the important questions raised in Mr. Little's valuable paper is that of the equity of the persistent as against that of the discontinuing policyholder. Mr. Little reaches the conclusion that the modern idea, in this country at least, is that the two should be placed on an equal footing, and that the withdrawing policyholder should receive the mathematical value of his policy, this value being determined not necessarily as the reserve by the mortality table, but by taking into account also the effect of surrenders upon future mortality.

The public has unquestionably demanded at least as much as this in surrender values, as shown both by statutes and its attitude towards different policy contracts. Whether, however, this expresses the real modern attitude towards the policy, or is merely the result of a misconception which will be transient in its nature, is not so clear. It is undoubtedly felt by the general public that the policy which is the most "liberal" in its provisions is the best for the policyholder. The public, however, does not comprehend the essence of participation in life insurance, which is that the policyholder receives in benefits all that the premium paid can purchase. Hence the public fails to realize that the addition of a "liberal" feature, which is an expense to the company, cuts off a corresponding benefit elsewhere, but on the contrary believes that the addition of the liberal feature means a total benefit that much greater.

I am not at all convinced, if the question could be put so that they would understand not only its words but also its meaning, that the insuring public would vote for the high surrender values. Some policies are taken with the deliberate intention of discontinuing after a certain number of years, but my belief is that the majority of applicants intend to carry their policies through to maturity, and would prefer an agreement that the discontinued policy should pay a reasonable surrender charge, provided the corresponding benefit goes to the persistent policies. This, however, is a question which at present there is no way of determining. If my belief is correct, and if in time a better understanding of the results of too "liberal" provisions comes about, there will be a reaction in opinion against very high surrender values.

Briefly, it is a case of trying to eat the cake and have it too, and when the public realizes it cannot do both, it will decide which it prefers.

A further point not to be overlooked is that high surrender values may hurt even the discontinuing policy more than they benefit it. If high surrender values—for instance the full reserve—mean many surrenders which are simply an exchange of a policy in one company for one in another, the policyholder who changes may not lose—may even gain—by his individual change, but the total result of the changes is the duplicating of initial expenses without increasing the total insurance in force and a corresponding decrease in total surplus from what it would have been had the changing policyholders simply kept their original policies. Thus, while the policyholder who changes in this way may not lose by his own change, he might be better off if all such changing could be stopped, because of larger surplus of which he would receive his share.

While the above may seem an argument against present surrender values, it is intended to be only a statement of the fact that I believe those who demand the high values are to a considerable extent doing so without understanding the bearing of their demand. On the one hand, if high values are really wanted there is no injustice in giving them and thus furnishing to the policyholder the sort of insurance he wants, even if this should go to the extent of discriminating against the persistent policyholder in favor of the one who surrenders. On the other hand, if the demand for high values is because of a misunderstanding, it is to be hoped that this is transient only, and that eventually the insuring public will come to demand what it really wants.

While the above applies directly only to participating policies, a similar argument applies indirectly to non-participating contracts. If a company promises all it can afford to in such contracts, there is a choice between a contract guaranteeing high surrender values and one guaranteeing another benefit of corresponding value.

The principal subject of the paper is the question whether surrenders increase the mortality; this I shall but touch on. Mr. Little has submitted strong evidence that they do, although unfortunately, this evidence has to be largely of an indirect or negative character. Mr. Hunter's paper presented at this meeting, showing the mortality following the end of the distribution period on deferred dividend policies, has added evidence which was lacking at the time Mr. Little wrote, and which tends to confirm Mr. Little's conclusion, at least in regard to one class of policies. It is true that Mr. Hunter's investigation refers to surrenders of a peculiar kind of policy, that is deferred dividend policies; and at a particular time, the end of the dividend period; and also that many policyholders took these policies with the idea of surrender-

ing at the end of the period. Hence, the evidence furnished by the investigation regarding these policies is not as strong, when applied to the effect of surrender under other conditions, as it might be. One of the conditions, however, present in these policies at the end of the deferred dividend period, will be continually present in policies now issued after they have been in force a few years, that is the condition of a high surrender value.

In the above I have not referred to the case of companies which have so suffered in public opinion that surrenders are made from doubt as to the future solvency.

MR. W. A. P. WOOD:

In his paper on "Modern Surrender Values" Mr. Little has directed our attention to a very important subject. We know of the gradual change that has been brought about on this continent in regard to amounts allowed for the cash surrender values of policies—these values having changed by degrees from very small amounts until at the present time it is the practice of most companies to allow the full reserve if the policy has been a few years in force. Mr. Little has intimated that at one time the companies took the stand that the assured had agreed to pay certain sums of money by way of premiums during the remainder of his life, or for a certain portion thereof in the case of limited payment life and endowment policies. If the assured wished to break this contract and surrender his policy they believed he was not entitled to much consideration and should pay at least a considerable penalty for his act. We have been shown how the more modern idea is to consider the policy as an alternative contract, the assured agreeing to pay premiums, not for life but only from year to year, and that, if at any time he wished to discontinue payment of premiums and surrender his policy, he should not be forced to pay a penalty for so doing. Partly on account of this opinion and partly, I am afraid, on account of competition, it has become the practice of most companies to allow a cash surrender value of the full reserve, except in the early policy years, where a small surrender charge is deducted.

The chief concern of the companies now is to know what cash surrender values they are justified in allowing, taking into account the rights of those policyholders who withdraw and of those who remain with the company. Mr. Little has mentioned the method of accumulating the premiums with interest and deducting expenses, dividends and cost of covering the risk (of death and deterioration). Whilst this method might be employed for the early policy years, I would not be in favor of making use of it to any great extent thereafter. One would have to make certain assumptions as to expense, rate of interest, mortality and allowance for deterioration of the risks. Thus in working up the funds, we would have quite a number of assumptions to make, and I doubt if,

in view of the changing conditions, the results obtained for the longer durations would be of great value. Then again I do not think the allowance for the cost of covering the risk of deterioration is very important. I am of the opinion that the companies would not be doing any great injustice to those continuing their policies by allowing the full reserve to those who withdraw after a period of, say, ten years. The exact duration, however, would be a matter for each company to settle for itself. From the paper under discussion, I would infer that Mr. Little considers an injustice is done to those who continue their policies, if those withdrawing are allowed the full reserve. He bases his opinion on the fact that he believes the mortality is greater amongst those who continue than amongst those who withdraw. The examples which he makes use of to prove this are in connection with British companies. It is quite possible that in Great Britain there might be a considerable difference between the mortality amongst those continuing and those withdrawing, and at the same time that the difference on this continent between these two classes would not be very great. I am of the opinion that in America this difference would be very small indeed, because I believe that financial conditions have a far greater effect in influencing policyholders to continue or withdraw than has the state of their health.

Suppose, for the present, we admit that the mortality experienced amongst those who withdraw is lighter than amongst those who continue their policies. We would still have to investigate and see if the mortality experienced amongst those remaining with the company is heavier than that assumed in the mortality experience on which the table of reserves is based. If the mortality among those remaining with the company is found to be less than that expected according to the table on which the reserves are based, I do not see that the giving of the full reserve to those who withdraw could be considered as doing an injustice to those continuing their policies, assuming, of course, that this practice is not followed in the early policy years, i. e., until any deficit in connection with the policy has been wiped out.

Mr. Little states that "those who have had occasion to consider the business of a company where the insurance in force is steadily diminishing will usually, if not invariably, have found heavy mortality prevailing." In this respect, I would like to refer to a section of the business of the company with which I am connected. This section deals with quinquennial dividend policies issued between the years 1846 and 1899, containing at the present time assurances amounting to about $36,000,000, with death claims running between $900,000 and $1,000,000 annually. Surely this business has been long enough in force to show any ill effects in the mortality that might be produced by the withdrawal of the more healthy lives. And yet, over a series of years, the mortality has been uniformly favorable. I find that the mortality experi-

enced has varied but little from year to year, and that in the past five years the actual cost of insurance has averaged about 72 per cent. of the expected according to the H^M Table of Mortality. This compares very favorably indeed with the mortality experienced in the entire business of the company, allowance being made for the effect of selection during the early policy years. Whilst I have not the experience of other companies in respect to the mortality on the old business, I believe that in most of our companies it will be well within the expected mortality.

Assuming that the mortality amongst those continuing their policies is slightly heavier than amongst those who have withdrawn, but lighter than that shown by the table on which the reserves are based, it follows that in the future there will still be a gain from favorable mortality, although not so large a gain as there would be if the other policies had not been surrendered. The gain from favorable mortality, however, would be as great as those continuing would be entitled to, and no hardship is caused on account of their dividends not being increased by the additional savings in mortality which would have materialized if the withdrawals had not taken place.

It is assumed in the above that the premiums charged are adequate. However, where a company only charges the net $3\frac{1}{2}$ per cent. premium with a very small loading and allows a cash surrender value of the full 3 per cent. reserve at the end of twenty years on life and limited payment life policies, it seems to me that it is going farther than is reasonable or justifiable. In some cases where the full 3 per cent. reserve is allowed as a cash surrender value, the gross premiums charged are not as great as the net 3 per cent. premiums, or even as large as the net $3\frac{1}{2}$ per cent. premiums required to provide for the $3\frac{1}{2}$ per cent. reserve and a pure endowment of the difference between the 3 per cent. and $3\frac{1}{2}$ per cent. reserves. I do not think that this practice can be justified. However, in the previous part of this discussion it has been assumed that all premiums charged were adequate.

As I understand it, the paper we are discussing deals only with the cash surrender values pertaining to new policies and is not intended to include such cases as the giving to old policies larger values than those inserted in the contracts. I do not think, in the case of deferred dividend policies, that the company would be justified in raising the surrender values in this way, as, in all probability, when the assured of this class were taking out their policies they were informed that the profits of those continuing their policies to the end of the dividend period would be increased by the amounts left with the company by those withdrawing from the fund. This has been held out to them as one of the special advantages of this kind of insurance, and the companies are bound to recognize this feature of their contracts.

During recent years there has been a strong tendency on the part

of the companies to deal more liberally with their policyholders. In fact, this liberality has become very pronounced, as shown by Mr. Moir in a recent paper read before the Society. It is quite possible that in some respects companies may have gone a little farther than is desirable in the best interest of life insurance. At the same time, however, as pointed out above, I cannot see, providing adequate premiums are charged, that the allowing of the full reserve as a cash surrender value on policies after they have been a number of years in force is depriving those who continue with the company of anything to which they are justly entitled, or that it is doing these policyholders any injustice.

MR. A. T. MACLEAN:

The question as to the proper allowance to be made on the surrender of a life policy is one which has been much debated, and Mr. Little has brought before us some of the fundamental considerations necessary to the solution of the question.

Mr. Little takes up, *inter alia,* two important aspects of the question of surrender values, first, the right of the policyholder to the full valuation reserve, and, second, the mortality amongst those who withdraw.

It has often been stated that the valuation reserve represents the individual share of the general fund, but I think that this view ignores the very principles of life insurance, and if there is a period in the history of a policy when it is desirable to know its true value, surely it is when this value is proposed to be withdrawn from the company.

The total of the individual reserves held by the company does represent the true present value of all its contracts when taken together. We know, however, that among these policyholders there are some at the point of death while others are in the best of health, and I do not think it requires even the mathematical demonstration given by Mr. Little to prove that if a healthy life withdraw and receive an average value, such policyholder is getting more than his true share. On the other hand (and assuming the true reserve to be the surrender value), a policyholder in bad health is not getting equitable treatment if he receive only the valuation reserve for the surrender of his policy.

We are, however, by reason of state laws and competition, practically forced to adopt the average reserve as the basis of our surrender values, so that the practical significance of the question is limited by these considerations.

If surrender values were based on the method suggested on page 262, we should probably find that it would be about five or six years on the average before any value could possibly be paid, and however justifiable this may be from the theoretical standpoint, the values produced by the method could not compete, at any rate in the early years, with the values given in modern contracts.

Mr. Little, assuming, on the whole, that the withdrawals will all be select lives, would in any event limit such values to the select reserve, unless "there is reason to suppose that the average prospects of longevity of the persons surrendering are no better than the average of those continuing their insurance."

Mr. Little states that we have "no inconsiderable body of circumstantial evidence" to prove that the mortality among persons surrendering is better than among the general body of policyholders, but I cannot see such evidence in his paper. I think the most liberal construction of what evidence is submitted, is that the matter is very much an open one upon which we have, so far, no real data to enable us to come to definite conclusions.

While it is true that selection is evidently exercised in the kind of insurance taken and also by annuitants, I do not think it follows therefrom that surrendering policyholders select against the company in withdrawing. From all I have been able to observe, the main factor in surrender seems to be the pecuniary condition of the insured. In the company with which I am connected there is a department devoted to the reinstatement of lapsed policies. This department endeavors to find out from the policyholder why the policy is discontinued, and I am informed that over 90 per cent. of the replies received indicate that the premium was unpaid simply through lack of funds. Further, in the case of the majority of those policies which are reinstated, the transaction is carried through by means of a loan from the company. I do not think it likely that people of this type are better than the average of those continuing.

Mr. Little quotes an experience based on those who took dividends in cash and those who took reversions. It seems to me, however, that the complete surrender of a policy and the surrender of the dividend are only to a certain extent analogous. The dividend is small in comparison with the entire value, and the motives underlying a surrender of the dividend are probably entirely different from those underlying the surrender of a policy. Again, in the experience cited, by far the larger part of the data consisted of cases where the selection of the option had been made once and for all at the date of the first dividend, so that, although selection may have been exercised as regards the first dividend, its effect on subsequent dividends—and these were quinquennial dividends—can hardly have been very great. Further, taking the cash dividend section, which covers almost as much experience as the other sections combined, we find that the ratio of actual to expected mortality among those who transferred from other options to the cash option, was 98 per cent., whereas the ratio among those originally taking cash was only 96 per cent. (*J. I. A.*, XXIII, 4). It would seem that selection should be much more pronounced in the case of a deliberate transfer, but the reverse appears to hold in this case, and instead of the mortality being lower in the transfer class, it

actually became higher. It is also shown in this investigation that the experience from age 60 onwards is practically identical under all the options, although it is just at these old ages that an effective selection might most be expected to appear. On the whole, however, the data is small, especially in the case of transfers, and for this reason alone, I think the experience can hardly be said to furnish very strong evidence on the point under discussion. In relation to present-day experience, also, the results are still further open to criticism when we remember that they are based on experience the very latest part of which is over thirty years old and derived from a period when life insurance methods were almost entirely different from those of to-day.

Mr. Little's criticism of Mr. Chatham's figures is, I think, a reasonable one to the extent that these figures do not prove or even indicate that withdrawals have an adverse effect on the rate of mortality. I cannot agree, however, that they "lead to a diametrically opposite conclusion," for the very reason given by Mr. Little, i. e., the lack of homogeneity in the data. This point was clearly brought out in the discussion on Mr. Chatham's paper.

The rate of withdrawal is heaviest just when the effect of selection is strongest and any attempt to investigate the effect of withdrawal on the rate of mortality is met with the difficulty of distinguishing between the two forces of medical selection and such withdrawal. In examining the following figures, therefore, these considerations must be kept in view. Table I gives the rates of mortality per 1,000 according to the recent Medico-Actuarial Investigation.

TABLE I.

Year.	Age at Issue.		
	22	32	42
1	3.3	3.7	4.7
2	4.5	4.8	6.4
3	4.6	4.9	6.9
4	4.7	5.0	7.5
5	4.8	5.2	8.0

Examination of these figures shows that the increase in the rates of mortality is about 30 per cent. between the first and second years and that the rates increase very slowly thereafter. It does not appear to me to be, *a priori*, likely that if the withdrawals were the better lives the mortality would show such a small rate of increase after the second year, but rather that it would increase rapidly for some years because of the increased proportion of inferior lives in the general body.

Table II gives the rates of mortality per 1,000, as shown in the O[M] Table.

TABLE II.

Year.	Age at Issue.		
	22	32	42
1	2.7	3.3	4.8
2	4.5	5.2	7.1
3	5.1	6.1	8.4
4	5.5	6.6	9.4
5	5.9	7.2	10.4

It will be noticed that in this case the increase in the rate of mortality between the first and second years is almost 70 per cent. at age 22 and 50 per cent at age 42. Further, the rate of increase in the other years is much larger than in the Medico-Actuarial Experience. I think it will be generally admitted that the rate of withdrawal in the Medico-Actuarial Experience was much higher than the rate experienced in the British investigation. In fact, in this particular section of the experience, the rate of withdrawal was exceedingly low, and at no age or duration was it as much as 10 per cent. (*J. I. A.*, XXXVII, 463). Nevertheless we see that the rate of mortality increases much more rapidly than in the Medico-Actuarial Experience. It may be noted that the medical selection was probably very similar in the two experiences, and this, of course, is a very important factor in considering the rates.

There are, of course, many differences between the two experiences quoted which make any definite conclusion from the above figures of very doubtful value. None the less I submit that the general indications they do give are against the theory that withdrawing lives are superior to the average.

There is one other point which might be noted, and that is the effect of the disability benefit which is now incorporated in most of our policy forms. Is it not probable that this will keep the inferior life with the company, and retain a contract on the books that in many cases would otherwise be surrendered? This benefit, in conjunction with the full reserve cash values mentioned by a previous speaker, must have the effect of creating a condition such as Mr. Little describes, and his paper, is therefore, all the more valuable in that it is a guide for the future as well as a review of the past.

MR. COBURN:

The following remarks in regard to the health of withdrawing lives may be of interest.

Mr. Besant, in the discussion that followed the reading of Mr. Fulford's paper at the Institute (*J. I. A.*, XXXV, 199), said that he had investigated the cause of 1,085 surrenders, and found that 840 or 77 per cent. of the whole were given up entirely from impecuniosity. This class of 77 per cent. exercises no option, and

it is hardly likely that their health is above the average. Mr. Besant was of opinion that they were below the average health. Mr. Chatham, following Mr. Besant, said that one of the methods of gauging the health of surrendering insured was to investigate the mortality on policies which had been purchased by reversionary companies instead of being surrendered. He believed that method had been adopted in one instance, and the result showed that the lives, which in the ordinary course would have withdrawn, were worse than those which remained It was quite true that in some cases the reversionary companies had information regarding the health of the insured, but Mr. Chatham understood that, as a rule, they had not.

The office for the combined investigation of mortality in German insurance companies has recently concluded the investigation into life insurances with years of entry 1876 to 1885. The observations on the insurances of those years of entry were continued up to 1905. Ungraduated select tables were prepared for each entry age and durations 1 to 30. Beginning with the eleventh year of insurance, the mortality for different attained ages varied considerably. For attained ages 40 to 50 the mortality remained nearly constant with the increased duration. For the higher attained ages the mortality curves rose considerably as the duration increased. For the lower attained ages the curves fell quickly as the duration increased.

Mr. G. W. Richmond contributed the following table to the *Insurance Record* of April 19, 1907:

TABLE II.

DEATHS AMONG THE WITHDRAWALS IN THE "SKANDIA" 1855 TO 1904, ASSURED AT ORDINARY RATES AND OBSERVED FROM THE DATE OF "EXIT."

Age Group.	Actual Deaths.	Expected Deaths per "Skandia" Experience.
20–24	7	5
25–29	21	18
30–34	44	34
35–39	73	46
40–44	76	60
45–49	83	79
50–54	86	93
55–59	88	77
60–64	72	70
65–69	61	56
70–74	43	40
75–79	19	26
80–84	7	8
85–89	4	3
20–89	684	615

These mortality curves were found to correspond well with population statistics, where the mortality was decreasing at the younger ages and slightly decreasing at middle ages.

The following opinion is given by Dr. A. Abel, who is in charge of this investigation:

"The self selection of healthy risks at time of termination or lapse can not have caused a decisive influence, because if that had been the case, the curves for the same observation ages for younger risks would have shown in later insurance years an increasing and not a decreasing tendency, as they actually do."

I submit that in normal practice the following conditions exist:

1. A large proportion of surrenders are due to impecuniosity.
2. The impecunious exercise no option on surrender.
3. Surrenders have no marked effect on mortality.

At the same time these conditions would be radically altered should there be a run on a company. Then only a small proportion of the withdrawals would be due to lack of means, an option would be exercised by the majority, and the mortality of the continuing insured would be increased.

ORAL DISCUSSION.

MR. HENDERSON: It has appeared to me on this question of withdrawal selection that the really disastrous selection which is exercised against the company is exercised, not by those who make up their minds that they do not want insurance any longer and are dropping their policies altogether, but by those who make up their minds that they can better themselves by dropping an old policy and either replacing it by a new policy in the same company or in another company, and in that connection a very important factor in the minds of a great many policyholders is the actual cost to those policyholders of the actual amount of insurance that they are receiving under their policies. The actual amount of insurance that they are receiving under their policies is the difference between the surrender value—the cash surrender value to which they are entitled—and the amount payable in case of death. The actual cost to them is the actual premium they are paying less any dividend received in reduction of that premium plus the interest which they suppose they can make if they draw cash value, or the interest they are paying if they borrow it plus the interest on the cash surrender value; and if a company is to protect itself against this kind of selection, it is important that its accumulation of contingency reserve and its dividend scale should be so adjusted that a policyholder cannot secure cheaper insurance, at least in that company, by dropping an old policy and taking a new one, because, if he can, there will be an accentuation of that kind of selection, whereby those who are eligible for new insurance will drop their old policies and put the company to the expense of placing a new policy on its books, and

those who are not eligible to new insurance will raise any money they need by a loan, or, in any event, will keep their policies in force.

MR. J. M. CRAIG: There was a time when the idea was pretty thoroughly imbedded in the actuarial mind that the reserve was a community as against an individual interest. We have been forced by competition and by law to break away from that idea, although it is theoretically sound, and just as sound today as it was twenty-five years ago, but if it is thought that because we are confronted with the legal phase of the question, a discussion as to the advisability of making a surrender larger or not is beside the question, I can hardly bring myself to agree. And for this very simple reason, that where the law prevails, according to my recollection, with one exception, and that is Louisiana, a charge is provided of 2½ per cent. of the face of the policy. Thus the law recognizes that it is fair to make a surrender charge up to that limit, so that it is a fair question of discussion what the surrender charge shall be within the limit of the full reserve, and the full reserve less 2½ per cent. of the face of the policy. And there is another phase of this question that ought to be kept in mind in the discussion and it is this: that while a company well established and able to conduct its business under a comparatively low rate of expense may be able to offer a surrender value with a charge of ten or fifteen dollars, or five dollars, or no dollars, there are other companies that are struggling for existence that can not afford to compete with such a company, and the law, so far as it regulates at all, regulates fully. I remember very distinctly when the Committee of Fifteen of the Insurance Commissioners were discussing this question that I wrote a letter to the Insurance Commissioner of Minnesota, who was the chairman of that committee, and called his attention to the fact that they were formulating laws not for the well-established companies only, but for the weak companies, and that it was the universal experience that the strong companies always were more liberal than the law provided, and that they ought to fix the surrender charge so high that a young company could live under it. I suggested 3 per cent. of the face of the policy so as to give the young companies a chance to live, and the country generally has seen that most companies can live under the legal aspect of the question if they are willing to take advantage of the full surrender charge that the law permits.

MR. WELCH: I can see, as Mr. Craig says, that a very strong company can do a great many things that companies much smaller and that are under a great deal more strain cannot afford to do, and I believe that one thing that we have got to do is to find out what surrender values cost the company. I do not apprehend any trouble in my company; I do not apprehend any trouble in a great many com-

panies that are today operating in the United States, on account of surrender values, but I think it is most important that we seek, as far as we can, to find out what effect withdrawals have. I think that the withdrawals in any of the larger and well-established companies, will not have any appreciable effect. They can give them without any surrender charge whatever. But I do not believe we can all do it. I know that we cannot without having trouble in the end. I am old enough to remember back when the reports came in of companies in trouble, of companies that went under, and of companies that were reinsured, and it is possible that another such period will come again after the birth of all the companies we have seen jump into existence within the past seven years—it must be so. And when any company loses that reputation for soundness and strength which is absolutely requisite for a life insurance company, much more so than for a bank, that moment the withdrawals from that company are going to exercise evil upon it that will be immeasurable, and we cannot afford to turn our backs upon that.

MR. IRELAND: I have just one word about the equities of this business. It is, of course, quite manifest that what is given in large and excessive values to retiring policyholders is so much taken from the funds and property of those who do not retire. It seems to me, in the present day, when almost every policy and every company set forth very clearly what they are going to do in the way of surrender values, that if one company says: "We will give large values to retiring holders and not quite so much to those who stay," and another company says: "We will be on the safe and prudent side and be friends of those who stay in, and not give so much to those who retire," no charge can be brought forward against any of those companies. They set forth in the policy what they are going to do, and it seems to me that within reasonable limits either of those contracts is a fair one.

MR. WELLS: I did not intend to speak but I feel that by silence I would be giving assent to something I did not believe in. Now, I do not believe in the new theory that the company's property,—reserve, if you choose,—is the property of the individual members of the company. It is community property and the policyholder is entitled to what his policy gives him and nothing more, nothing less. We are under obligation, as law-abiding citizens, to make our contracts conform to the law, but what belongs to the policyholder belongs to him, not as a matter of right, but because the policy gives it to him, because the contract gives it to him.

There is one other matter I want to speak of—as regards the possibility of a variation in the mortality as between those who withdraw and those who do not withdraw. The experience of years past has been referred to here in the reference to Mr. Chatham's paper,

but we are not living in those days. Our policies, probably all of them, provide for a loan, for instance. No man is obliged by pecuniary conditions to drop his policy. He can drop it or get a loan. I say "No man." Perhaps I ought to except the first three years of the life of the policy, but in general, a man who has got a policy can get a loan on it from the company if he chooses to pay his premium. He is not, in general, obliged to drop his policy for pecuniary reasons. If he drops it he drops it for some other reason. Perhaps, in the past, it was true that a man dropping his policy for pecuniary reasons had to drop it whether or no, but that is not true in the same sense today.

MR. HUNTINGTON: If a new company or a weak company starts out to sell an inferior policy in order to protect itself, it is handicapped from the start, and it is a question if you do not save in the spigot and lose at the bung. In order to conduct your business economically you must have a policy that will run as well in surrender values as your next-door neighbor who is strong, and you are going to handicap your agents, if you do not make a policy that on its face will bear comparison with any other policy. It is a question whether it is worth while for a new and weak company to try to protect itself in that particular way and be certain to make a loss rather than provide itself with some sort of protection.

MR. MACAULAY: I would like to say in connection with what Mr. Huntington has said, that there is a limit in regard to surrender values beyond which a new company of the kind he has in mind cannot go. Those companies have the preliminary term system of valuation. Such a company cannot afford to give any surrender value that would be larger than the reserve on a policy according to its system of valuation.

MR. LITTLE.

(AUTHOR'S REVIEW OF DISCUSSIONS.)

It is gratifying to find that the main principle of my paper has met with little criticism, it being, apparently, generally accepted that the ideal surrender value constitutes a return to the outgoing policyholder of the excess of his payments over all items of disbursement properly chargeable against his policy.

The chief discussion has centered on the theory, maintained by me, that surrendering policyholders are, on the average, better lives than those not surrendering of corresponding age and duration of insurance. This theory has by no means met with general acceptance, and it is therefore desirable to deal with the criticisms thereof in some detail. These criticisms may conveniently be divided into three sections, (*a*) those directed at the evidence submitted by me, (*b*) those introducing fresh evidence apparently unfavorable to the

theory in question, and (*c*) criticisms based on general reasoning. These will be dealt with in order and thereafter some new evidence, not available when the paper was written, will be referred to.

Mr. Moir and Mr. Maclean have questioned my conclusions arrived at from a study of Mr. Chatham's figures. Both gentlemen, however, appear to think I attach more importance to these figures than I do. It is true I have stated that they "lead to a diametrically opposite conclusion" to that of Mr. Chatham, but I have carefully pointed out that the comparison is of very doubtful validity. In other words, I stated that the figures were unsafe as a guide, but added that if any were nevertheless disposed to accept their guidance, they would "lead" to a conclusion favorable to the opinion I was advancing, and this appears to be admitted by both Mr. Moir and Mr. Maclean.

I am afraid I cannot follow Mr. Maclean's criticisms with reference to the experience of the British Empire Mutual under different forms of dividend options. Mr. Maclean states that "the motives underlying the surrender of a dividend are probably entirely different from those underlying the surrender of a policy," and also expresses the opinion that "the main factor in surrender seems to be the pecuniary condition of the insured." If this is to be taken to mean that the pecuniary condition of the insured is unimportant in its influence on the choice of a dividend, I cannot agree with Mr. Maclean. It should be noted that Mr. G. F. Hardy, in presenting the experience, expressed an opinion that the selection of a cash dividend was made from motives similar to those causing surrenders.

Mr. Maclean goes on to point out that the dividend system is selected once for all (except for transfers) at the date of first dividend, but this is surely strictly analogous to the case of surrender, which also happens only once in the history of a policy. This fact also explains the feature that from age 60 onwards no effect of selection is apparent, as at these ages the mortality results would be almost wholly contributed by insured of long standing, and we should not, therefore, suppose, as claimed by Mr. Maclean, that "it is just at these old ages that an effective selection might be most expected to appear." Mr. Maclean also points out that this experience is old, but I am inclined to think that the proper deduction from this fact is that the amount of selection under similar circumstances today would be greater, the insuring public being more conversant with the business.

Coming next to the fresh facts presented by various speakers, Mr. Coburn has submitted two very interesting statements as applying to the question under consideration. The first is taken from a recent German experience in which it was found that for ages attained under 40, the mortality at a given attained age did not rise with increased duration of the insurance after the tenth policy year. From a graduated select table appearing in *Deutsche Ver-*

sicherungs Zeitung of April 6 last, for the attained ages in question the mortality seems to reach a maximum, for a constant attained age, about the seventh policy year. This condition Mr. Coburn apparently regards as indicating that policyholders surrendering at these younger ages were no better, if not worse, lives than the average. To state the contention simply, the argument is that at age attained 35, persons entering at age 25 showed a higher rate of mortality than those entering at age 24, although the latter had one additional year's prior duration, and had, therefore, been one year longer exposed to the effect produced on mortality by withdrawals, indicating that these latter did not adversely affect the mortality. This, at first sight, seems plausible and, therefore, merits careful examination.

In the first place I must point out that, assuming withdrawals to be uniformly better lives than the average, according to age and duration, it does not follow that the maximum effect is produced at the longest duration. A simple illustration will make this clear. Suppose there are 1,000 lives at the beginning of a year, of whom 100 are very bad, and subject to a death rate of 25 per cent. per annum. The 100 bad lives will thus be reduced to 75 at the end of the year, and if, of the 900 remaining lives, 1 per cent. die, 1 per cent. become very bad lives and 2 per cent. surrender, the total of the very bad lives at the end of the year will be $75 + 9 = 84$, and of the remaining lives $900 - 9 - 9 - 18 = 864$. The very bad lives thus constitute 9 per cent. of the total, instead of 10 per cent., as at the beginning of the year, although the withdrawals were better than average lives, as they included none of the very bad lives. It follows that insured lives reaching age 28, say, who entered at age 20, *may* include a smaller proportion of bad lives than those of the same attained age who entered at age 21. In such case the eighth policy year would show lower mortality than the seventh, notwithstanding the fact that withdrawals in all policy years had uniformly been better than average lives.

Secondly, I would point out that in Germany whole life policies and endowments practically constitute the whole of the issues. At the young ages endowments predominate, and as the entry age increases, the proportion of endowments steadily falls, and of whole life insurances rises. At the starting point, therefore, the younger the age the greater the proportion of superior lives, and it follows that, using the ages named above, lives reaching age 28 of entrants at age 20 will be taken from a somewhat better class than those of entrants at age 21, and would show slightly lower mortality in the absence of any other feature affecting the results. Neither the effect of this factor, however, nor the maximum result of the condition previously discussed, would account for the *rapid* fall referred to by Mr. Coburn. We are thus compelled to seek for some more potent cause.

Thirdly, and lastly, attention must be called to the method adopted

in constructing the mortality table in question. The issues observed were those of the years 1876–1885, and the first policy year's experience was consequently included within the calendar years 1876–1886, the experience of the second policy year within 1877–1887, that of the third policy year within 1878–1888, and so on. The experience of each policy was, therefore, chronologically, just one year later than that of the preceding policy year. If, now, mortality were generally decreasing during the period of observation, diminishing mortality, with increasing policy duration at a given age attained, would follow in the absence of any more powerful counteracting influence. The following is an extract from the second volume of the *Proceedings of the Seventh International Congress of Actuaries,* page 12:

GRADUATED RATE OF MORTALITY PER THOUSAND LIVING.
GERMAN EMPIRE—MALES.

Ages.	Rate of Mortality Deduced from Statistics of			Percentage Decrease in Rate of Mortality	
	1871–1881	1881–1890	1891–1900	From 1871–1881 to 1881–1890.	From 1881–1890 to 1891–1900.
20–30	8.47	7.21	5.96	15.0	17.3
30–40	10.95	10.27	8.25	6.2	19.7
40–50	16.67	15.65	13.97	6.1	10.7
50–60	27.72	26.32	24.32	5.1	7.6
60–70	54.61	51.75	48.94	5.2	5.4
70–80	118.38	114.44	108.81	3.3	4.9

I desire to call attention to the remarkable manner in which the above figures accord with the results referred to by Mr. Coburn. The surprising decrease in mortality at ages under 40 coincides with decreased mortality (at a given attained age) as policy duration increased; the smaller, but still considerable, decrease at ages 40–50, is responded to by mortality level for various durations; while the lower decrease at ages above 50 has proved insufficient to offset some other cause operating in the opposite direction. I feel I am justified in asserting, therefore, that the facts noted concerning attained ages under 40 are clearly explicable on the assumption of superior average vitality of surrendering policyholders, while the conditions shown at ages over 40 are difficult to explain except on such assumption.

I do not know how the experience of the surrendering policyholders of the "Skandia" company was obtained, but, assuming the statistics to be accurate, some very interesting results may be obtained. It may, doubtless, be taken for granted that the "Expected deaths per 'Skandia' experience" were calculated by an aggregate table, and, if so, another instance is provided of the misleading results of comparisons based on aggregate tables. The

experience of these surrendering policyholders, after surrender, is analogous to that of an annuity experience, in which there are no withdrawals, and which consequently includes a much greater proportion of the exposures of the later policy years than does an experience of insured lives, where the heavy withdrawals in the early years reduce the relative amount of exposure contributed by the later insurance years. A simple illustration may make the matter clear. On page 94 of Vol. I of the Report of the Medico-Actuarial Investigation it is shown that there were 35,273 entrants at ages 20–24, with an average age of 22 years, of whom 18,567 were exposed in the sixth policy year, corresponding to age 27. New entrants at average age 27 were 51,397, and if we suppose the deaths agreed with the M. A. Table, we should have:

18,567	exposed at age 27 in sixth policy year resulting in	89 deaths
51,397	exposed at age 27 in first policy year resulting in	180 deaths
69,964	exposed in all, resulting in	269 deaths

and giving a mortality rate of .00384. If, however, the 35,273 entrants as above had been reduced by death only, there would have been 34,508 exposed in the sixth policy year, with 166 deaths, adding to which the new entrants, the total exposures become 85,905 and the total deaths 346. The expected deaths by the aggregate rate formed in the ordinary way are 330, the "actual" being 5 per cent. in excess. Were the entrants from earlier ages brought into account, the disturbance would be increased. The excess shown above is purely artificial, identical mortality rates having been assumed to prevail for the two groupings. It is important to notice that if the excess of 11 per cent. shown by the surrendering policyholders in the "Skandia" were explained in this manner, it would indicate that those who surrendered were exactly equal to newly selected lives, and it is clear that, in any case, they were very little inferior, and undoubtedly better than remaining policyholders of similar durations and ages.

Coming now to Mr. Maclean's comparison of the O[M] with the M.A. Table, and the more rapid rise in mortality shown by the former, it is to be observed that in the O[M] Table the exposures of all policy years alike were derived from observations of the calendar years 1863 to 1893, and the effect on the apparent degree of selection of generally improving mortality was negligible. In the case of the M.A. Table, the mortality of the first policy year is drawn from observations of the calendar years 1885 to 1909, the second policy year's experience is of observations of 1886 to 1909, the third, of observations of 1887 to 1909. Each policy year's experience was, therefore, chronologically, more recent than the preceding one, whence the effect of the improving mortality shown to have existed has been to diminish the calculated increase in mortality with duration from issue.

Secondly, the medical examinations for issues entering into the O[M] experience were made in 1892 and during more than half a century previously; those affecting the M.A. Table were made from 1885 to 1908. I think few will agree with Mr. Maclean that "the medical selection was probably very similar in the two experiences," the fact being that medical selection has unquestionably become more efficient in recent years.

A third point of difference may lie in the relative extent to which "dating back" has affected the two experiences. With this unknown, the ratio between the mortality of the first policy year and that of the second is not a comparable item. Another point is that the O[M] Table is based on ordinary whole life policies only, while the M.A. experience covers all usual plans. The selection against the company is not at all likely to be uniform over various plans of insurance.

Yet another difference arises from the complete absence of the extended insurance option from the policies entering into the British experience. Recently published statistics have shown that this feature must have caused the exclusion from the M.A. experience of many deaths, which would otherwise have been included, the incidence of such deaths falling more heavily with increasing duration. The totally different conditions affecting the two tables sufficiently account for the differences noted by Mr. Maclean.

Mr. W. A. P. Wood refers to my remark that, where the insurance in force in a company is diminishing, heavy mortality will usually prevail. Mr. Wood mentions the experience of a closed section of business in an active company, but what was in my mind was that a company showing a decline was apt to have its business raided by the less scrupulous of the agents of its rivals, and policies "transferred" under such circumstances must necessarily be on select lives. Mr. Wood appears to consider the very favorable mortality of the section in question as negativing the suggestion that it has been adversely affected by surrenders. The favorable result, however, is chiefly indicative of the low mortality prevailing in Canada, while the fact that one section of a company's business is better than another does not prove that the mortality in both sections might not have been better under different circumstances.

Mr. Coburn quotes the results of an investigation into causes of surrender, showing that out of 1,085 cases, 840, or 77 per cent., surrendered owing to impecuniosity. Mr. Coburn adds that "this class of 77 per cent. exercise no option." I would suggest, however, that the true history of this class may be somewhat as follows: "900 policyholders became impecunious and 840 surrendered their policies, but the remaining 60, finding their health seriously impaired, borrowed upon their policies instead of surrendering." Clearly, to know whether the impecunious exercise an option, it is necessary to know the whole number of them, and not only those who surrendered.

Mr. Coburn also refers to a case mentioned in my paper, where a reversionary company purchased a number of policies on bad lives, and a further number on average lives, and found, naturally enough, that the mortality was above the average. Surely no comment is necessary upon a proposal to gauge the health of *withdrawing* policyholders by that of group of lives, partly selected because of known ill-health, and which *did not withdraw.*

This completes the review of the evidence supposedly adverse to the theory that surrendering policyholders are better than average lives. Not a single apparently adverse point has stood the test of critical examination, and the theory remains unscathed.

On general reasoning the chief adverse argument is that persons who surrender do so in most cases because of impecuniosity. This is probably true, but, as hinted above, a person suffering from an incurable malady will borrow rather than surrender. It may make my position clearer if we suppose the insured divided into three classes—the specially good lives, the specially bad lives, and the intervening group. Mr. Moir regards the first section as the thrifty lives who do not surrender. But such lives *do* surrender at the insistence of an agent to join a supposedly better company. In these regrettable transactions, if the life in question is passed "select," the old policy is dropped, but if the life is rejected the old policy is continued. From this special cause the best of the lives are liable to surrender, and surrenders also take place in the middle section. In the worst section, however, surrenders are rare, and this is the section that will, during the few years next succeeding, provide a disproportionate part of the death claims. As compared with Mr. Moir's personal experience, I may mention that I cannot remember in a long experience a really bad life surrendering, but knew of several instances of relatives keeping such policies on foot.

As a last word on this subject I would call attention to Mr. Rhodes's paper on the mortality experience on paid-up policies in the last number of the *Transactions,* and to Mr. Hunter's paper in this number. Mr. Gore has pointed out that we have a mortality experience in the former closely analogous to that of surrendering policyholders, the prompting motives being undoubtedly similar, and a severe adverse selection was shown. Just as striking, as pointed out by Mr. Strong, are the results of Mr. Hunter's investigation into the mortality of policyholders electing to continue their policies after the expiration of deferred dividend periods. These two experiences are emphatically favorable to the theory of the average good health of surrendering policyholders.

Some remaining points I cannot allow to go by default. Mr. Moir disagrees with my statement that decidedly bad lives practically never surrender, but I have no doubt he agrees with my intention. The knowledge that *some* bad lives surrender does not affect my statement. I agree with Mr. Moir, also, that the con-

tingency reserve must be available in case of necessity, however arising, but I still think its proper sphere is as I have outlined it. I am inclined to think Mr. Moir is right in supposing that the tendency in recent years has been away from the opinion that surrendered policies were on good lives. For this reason I marshalled the available evidence, and have replied at length to criticisms, believing that I was to some extent substituting "facts for appearances and demonstrations for impressions."

I do not know how Mr. Maclean arrived at the supposition that the method of determining surrender values advocated by me would prevent any being paid till after five or six years' duration. As a matter of fact there is nothing in the principle to exclude payment after one year, though in most cases two or three years would be required in practice. I must point out, also, that I do not assume that withdrawals will all be select lives, as Mr. Maclean states. Mr. Wood infers that I consider an injustice done if those who withdraw receive the full reserve. A careful reading of the paper will show that I see no objection to paying the full reserve if the company is in some way reimbursed for its full outlay. Dividend systems frequently render payment of the full reserve to holders of old policies no more than justice.

Mr. Wood appears to consider that if the valuation table allows for a mortality in excess of that experienced, there is no need to adjust surrender values on account of increased mortality resulting from withdrawal. This is another way of saying that, as the premium charged continuing members is more than enough, instead of returning the whole excess, some part may be applied to make good the loss experienced by charging outgoing policyholders with less than the actual cost of the risk run in their behalf. I do not think the surplus belonging to continuing members should be used for this purpose.

Mr. Strong expresses the opinion that as high surrender values are given in response to demand on the part of the public, even if they discriminate against continuing policyholders no injustice is done, as the kind of protection asked for is furnished. Mr. Ireland expressed a somewhat similar view, holding that even considerable divergences in surrender values were justifiable as being part of the contract, understood before being entered into. I am not prepared to dispute the logic underlying this aspect, but I feel that the better course is to take the view that, as the policyholders are so relatively ignorant of the whole matter, the companies should regard themselves as trustees, bound to offer the fairest possible contract, and to disregard a demand, arising from ignorance, for conditions really unfavorable to the insured.

A NEW ANNUITY EXPERIENCE—JOHN S. THOMPSON.

VOL. XIII, PAGE 273.

WRITTEN DISCUSSION.

MR. RHODES:

The presentation by Mr. Thompson of the annuity experience of the Mutual Life Insurance Company is very welcome for several reasons, but especially so, to my mind, because it affords us an opportunity to form an intelligent opinion regarding the sufficiency of the rates now in use when applied to business in the United States and Canada. At the ages at which annuities are most frequently issued the Mutual Life experience shows rates of mortality considerably under those shown by the McClintock Table, which is now the generally recognized standard for valuation purposes. The companies have a margin in the excess interest earned upon the invested reserves, but it remains to be demonstrated whether the gains from excess interest offset the losses arising from low mortality. The successive changes made in annuity rates of late years, all of which, so far as I recall, have been in the direction of an increase, indicate very clearly that annuity business in the United States and Canada has not been entirely satisfactory. I have computed the net premiums based upon rates of mortality equal to 60 per cent. of those shown by the McClintock Table for male and female lives, and others, on the basis of 80 per cent. of the McClintock Table. Interest is assumed at the rate of 4½ per cent., which corresponds fairly well with the average net rate now being earned on the invested reserves. The net premiums so computed are compared in the following tables with the gross rates now charged by a number of large companies. The percentage which these net premiums bear to the gross rates is also shown.

MALE LIVES.

Age.	Gross Rates.	Net Premiums.			
		60 Per Cent. McClintock and 4½ Per Cent. Interest.	Per Cent. of Gross Rate.	80 Per Cent. McClintock and 4½ Per Cent. Interest.	Per Cent. of Gross Rate.
51	$14.865	$13.782	92.71	$12.782	85.99
55	13.420	12.793	95.33	11.752	87.57
60	11.577	11.438	98.80	10.356	89.45
65	9.752	9.979	102.33	8.878	91.04
70	8.011	8.464	105.65	7.373	92.04
75	6.516	6.953	106.71	5.907	90.65
80	5.468	5.515	100.86	4.547	83.16
85	4.895	4.217	86.15	3.351	68.46

FEMALE LIVES.

Age.	Gross Rates.	Net Premiums.			
		60 Per Cent. McClintock and 4½ Per Cent. Interest.	Per Cent. of Gross Rate.	80 Per Cent. McClintock and 4½ Per Cent. Interest.	Per Cent. of Gross Rate.
51	$15.769	$14.882	94.38	$13.990	88.72
55	14.472	13.908	96.10	12.955	89.52
60	12.675	12.562	99.11	11.543	91.07
65	10.708	11.096	103.62	10.029	93.66
70	8.783	9.550	108.73	8.463	96.36
75	7.289	7.978	109.45	6.907	94.76
80	6.035	6.447	106.83	5.430	89.98
85	5.048	5.027	99.58	5.097	81.16

It will be seen that, if the mortality be as low as 60 per cent. of the McClintock Table, and if interest earnings be equal to 4½ per cent., the present rates are insufficient. If the mortality be equal to 80 per cent. of the McClintock Table, and if interest earnings be 4½ per cent., the present rates, after deducting commissions and other expenses, are sufficient for male lives, and, in some cases, insufficient for female lives. It is clear that the companies cannot count upon mortality equal to that of the McClintock Table, and it is noticeable that in the Mutual Life experience the divergence from that table is greater in the case of women than of men. This is shown in the following table, which shows the ratio of the rates of mortality of the Mutual Life experience to those of the McClintock Table:

Age.	Male Lives.	Female Lives.
40	1.15	.64
50	1.02	.71
60	.90	.76
70	.82	.80
80	.78	.82

ORAL DISCUSSION.

MR. HENDERSON: There were just a few points in connection with the paper which are perhaps worth noting. I took the trouble to reproduce the A and B constants which Mr. Thompson must have used and there was a very interesting feature of those four sets of constants which seemed to me suggestive. The point in connection with it was that in comparing the domestic with foreign of the males, the constants varied in the same way as in comparing the same two subdivisions of the females, and in the domestic in comparing male and female the constants varied the same way as they did in the foreign. That is, the males showed a very high A'

constant in domestic and foreign, and the females a very low one, and the *B* constant in the domestic was much less in both male and female than it was in the foreign. I did not have time to come to any definite conclusion as to what was the physical reason behind that, but I thought I would suggest the point to the Society. Of course, the practical importance of Mr. Thompson's paper is in confirming the results of Mr. Hunter's investigation. It showed, at least so far as the Mutual was concerned, that the data included in Mr. Hunter's investigation were homogeneous—that they coincided, so far as the average in the United States was concerned, with the general average of all the companies included, and I think that all the other large companies would individually confirm the same thing. It indicates that Mr. McClintock's Table is a substantial representation of the average of the Mutual's experience, but it is not a representation of the probable experience of annuities sold in this country under conditions such as have prevailed in the past.

MR. THOMPSON:

(AUTHOR'S REVIEW OF DISCUSSIONS.)

Although the intention in submitting this experience was to furnish some hitherto unpublished comparison of foreign and domestic annuitants as shown in the business of the same company, the most interesting observations have so far been drawn from the study of the results shown by the domestic business alone. The question of reserves only was taken up in the paper, because the use of aggregate annuity factors in a test valuation corresponded to the general use of the McClintock Annuity Tables in actual valuations. The question of the sufficiency of the companies' present schedules of rates is, if possible, of even greater importance. Mr. Rhodes has presented facts bearing on this point which emphasize the necessity of a liberal interest margin over the valuation rate to compensate for the generally higher mortality shown in the tables which are used in the calculation of rates. In a discussion of Mr. Hunter's report (*T. A. S. A.*, XII, 261) on the combined experience of American life annuitants, Mr. Gore expresses the opinion that an interest margin of two thirds of 1 per cent. should be maintained in order to cover the improvement in the mortality of our annuitants over the mortality among the annuitants of British Offices. The gross premiums used by Mr. Rhodes appear, after an examination of one of the well-known rate manuals, to be among the highest quoted for life annuities. On the other hand, the rates of mortality shown by the Mutual's experience appear to be more adverse than those experienced by the other companies of Mr. Hunter's combined experience. This conclusion is based on an observation of the general increase in the rates of mortality shown by the combined experience of 1910 over that of 1904, when

the expected deaths are computed by the British Offices' Experience; the inference that a similar experience would be shown if the expected deaths were calculated by using the McClintock Tables, leads to the conclusion that the ratio of actual to expected deaths would exceed respectively 82 per cent., the Mutual's ratio of actual to expected in the case of male lives, and 79 per cent., the Mutual's ratio of actual to expected in the case of female lives, the difference being much more marked in the case of the latter. This difference between the rates of mortality experienced by those companies which use high rates and those which use lower rates for life annuities may be due to the fact that if a person purchases a policy at a high rate of premium the selection will be made in the most forcible manner, but that if an estate is required to set up a life income for a legatee, the contract will probably be placed in that company which gives the best return on the money invested. The existence of two such classes of purchasers of annuity contracts is pointed out by Mr. Gore in another connection.

There is one rather important point which should not be overlooked, namely, that the single premiums for life annuities calculated by means of select tables are less than those calculated by an aggregate table, at the younger ages, and are higher at the older ages. In other words, at the ages used by Mr. Rhodes in his comparison, which are those below which contracts are seldom written, the select rates would be even larger, and the interest margin required correspondingly greater.

A THEORY OF SUB-STANDARD LIVES—ALBERT W. WHITNEY.

VOL. XIII, PAGE 282.

WRITTEN DISCUSSION.

MR. KILGOUR:

There is perhaps no subject in the whole realm of actuarial science that is more inviting to the student of abstruse mathematics, as well as to the practical actuary, than that of discovering a curve and evolving a mathematical formula that will fit in with the factors underlying the table of mortality. Their application to sub-standard lives, which is somewhat new, affords a most interesting study, and it is to be hoped that the present paper will open up a fruitful field for future investigation and experiment.

The well-known formulas of Gompertz and Makeham are admirable attempts to harmonize a theory with the facts at our disposal, while a number of more involved, but less practical formulas, have been adduced for the same purpose. The most superficial student, however, is aware of the fact that they are mere attempts to express in a rather rough and crude way the underlying forces operating in connection with human life, and while we accept these formulas and the resultant laws for practical purposes, we must admit their limitations and be extremely careful in attempting to apply them in cases where there are liable to be important deviations from the standard or the average cases.

In appraising a risk, there are many factors which require consideration, such as domicile, habits, environment, moral hazard, physical fitness, personal history of disease, family history and heredity. The circumstance that these factors, when taken in conjuction, admit of the application of a simple law, does not imply that a similar law can be utilized in appraising the individual factors and, much less, important deviations from what we may term the average individual factors. At the present time we certainly have not any statistics at our disposal to prove that this can be done, and I doubt very much if it will ever be possible to collate our statistics in such a way that they can be translated into such simple factors as those involved in what Mr. Whitney terms a "parametric method" of rating sub-standard lives.

All pathological states can be roughly classified as due to (*a*) bacterial infection, (*b*) internal disturbances to nutrition, (*c*) animal parasites, (*d*) injury from outside sources, (*e*) any combination of the above. To these should be added heredity, the effect of which is most difficult to determine and almost impossible to appraise in the

individual cases. It is outside of the scope of a review of this nature to enter into any discussion of these various factors and the way in which they influence the individual risk. I think, however, I am safe in stating that, in the present state of transition in medical science and with the constant discoveries being made, it is unsafe to dogmatize, and almost impossible to formulate theories that are likely to stand the test of time. The practical actuary who, with the medical director, has to pass on the company's applications, will appreciate the extreme difficulty of reducing to intelligible mathematical factors the theories and principles underlying the science of medicine, and must admit that, for the present at least, we shall have to remain content with fairly broad classifications.

In the recent Medico-Actuarial Investigation there were about one hundred classes dealt with under the headings, "Personal History," "Family History" and "Medical Impairments." The effects of build and occupation were also considered. It will be extremely interesting, and I would like the author of this paper to consider, with the statistics soon available, the practical application of his method of treating sub-standard lives in these various classes. I question very much whether he will find on investigation that Makeham's law is applicable to any more than a very few of them. There is every reason, I believe, to suppose that γ is not a constant factor in the majority of classes involving impaired risks. As stated by Mr. Whitney, the impairment of the constitution by disease means not only the increase of β, but also the increase of γ, which at the moment of death from disease becomes infinite. If, on the other hand, we admit, for the sake of argument, that Makeham's law may be supposed reasonably to apply and that we have a sufficient body of statistics to determine the α, β and γ for the necessary classes, we still always have with us the difficulty of measuring in the individual cases the importance of the suspected deviation from the average of the class. Will it be an easy matter for either the actuary or the medical director, or both together, to state approximately in what way this deviation will affect the constants of Makeham's formula?

The interdependence of the factors is also a point which the author alludes to, and one which must create great difficulty in dealing with the method practically.

In treating with impaired risks the collateral factors which are, in the standard cases, often ignored, must always be taken into consideration and necessarily influence the rating of the risk. Unless we are content with the broadest classifications, I think we may as safely leave this, as we do at the present time, to what may be called the enlightened intuition of the medical director and the actuary rather than the more refined system involved in the method proposed by Mr. Whitney. There are, of course, obviously, many classes, such as occupational, and a few involving well recognized and appraisable medical impairments, to which this method may be

made to apply, but that it may be generally utilized, I very seriously doubt.

Much might be said in connection with the determination of the rates and the reserves and other values used in standard policies. Will it be practicable to calculate exactly the appropriate reserves for all the plans for standard variations of α, β and γ? The work involved, at least, will be extremely great.

Mr. Whitney's mathematical demonstration is clear and straightforward. It occurred to me that it would have been preferable to make use of the more familiar symbols of Makeham's formula and determine the conditions given in formulas (11) and (12) as follows:

$$l_x = ks^x g^{c^x} \quad \text{Makeham's formula;}$$

$$\therefore \log_e l_x = \log_e k + x \log_e s + c^x \log_e g$$

and

$$\mu_x = -\frac{d \log_e l_x}{dx} = -\log_e s - \log_e g \log_e c \cdot c^x = \alpha + \beta e^{\gamma x}$$

where

$$\alpha = -\log_e s, \quad \beta = -\log_e g \log_e c$$

and

$$\gamma = \log_e c.$$

Also

$$\log_e l_{x+t} = \log_e k + (x+t) \log_e s + c^{x+t} \log_e g.$$

$$\therefore \log_e {}_tp_x = t \log_e s + c^x (c^t - 1) \log_e g$$

$$= -\alpha t - \frac{\beta e^{\gamma x}}{\gamma} (e^{\gamma t} - 1).$$

Now

$$\bar{a}_x = \int_0^\infty v^t \cdot {}_tp_x \cdot dt = \int_0^\infty e^{-\delta t} e^{-\delta t - \frac{\beta e^{\gamma x}}{\gamma}(e^{\gamma t}-1)} \cdot dt.$$

since

$$e^{-\delta} = v$$

$$= \frac{e^{\frac{\beta e^{\gamma x}}{\gamma}}}{\gamma} \int_0^\infty e^{-\left\{+\left(\frac{\alpha+\delta}{\gamma}+1\right)\log_e z + \frac{\beta e^{\gamma x}}{\gamma} \cdot z\right\}} \cdot dz$$

When we put

$$e^{\gamma t} = z.$$

$\therefore \gamma \bar{a} = \gamma' \bar{a}'$ for all values of α, β, γ, δ and x.
where

$$\frac{\alpha + \delta}{\gamma} + 1 = \frac{\alpha' + \delta'}{\gamma'} + 1$$

and

$$\frac{\beta}{\gamma} e^{\gamma x} = \frac{\beta'}{\gamma'} e^{\gamma' x'}.$$

MR. LITTLE.

The principle underlying Professor Whitney's theory of sub-standard lives is that Makeham's hypothesis is a true, or closely approximate, expression of the law of mortality at adult ages under all circumstances. It will generally be admitted that for standard lives, after the effect of selection has disappeared, the hypothesis gives results in close agreement with fact, at least for male lives. There is, however, from the nature of the case no very definite *a priori* theory upon which the formula is based, and it has attained deserved prominence only because practical tests have proved its fitness.

In applying the same formula to sub-standard lives, it is evident that the only measure of its dependability is actual test in practice, and it will be of value to examine Professor Whitney's suggestion from this point of view.

Professor Whitney takes the familiar expression for the force of mortality

$$A + Bc^x$$

and points out that any one or more of the three constants may be increased to meet sub-standard conditions. The first of these constants, A, is described as representing the force of mortality due to environment, occupation, accident and infection, or some parts thereof. The second constant, B, represents the growing portion of the force of mortality (or, rather, at age x, $B_x = Bc^x$, represents the attained extent of such growing portion) arising from tendencies inherited or acquired. The third constant, c, is the rate at which the tendency to death is growing, though Professor Whitney applies this description to γ, the Napierian logarithm of c. This, it is stated, "may depend partly upon effect of habits and environment and partly upon heredity and past life." It will be noticed that this last phrase is sufficiently comprehensive to include every possible source of influence upon mortality, and with such definition I am inclined to agree.

Subsequently, however, Professor Whitney expresses the opinion that γ must be closely connected with habits, the inference being that bad habits may fairly be allowed for by an increase in γ, or, what is exactly the same thing, an increase in c. The result of increasing c is to add to the force of mortality a percentage growing very rapidly with age, with the result that the excess mortality is represented as being almost wholly much deferred.

With this view I cannot agree. It seems to me that if we had under observation today a group of persons all guilty of bad habits,

that the next ten years' experience would show a very high mortality, and that the survivors, some of whom would have reformed, and who would generally consist of those upon whom the habits had least effect, while still a poor class of lives, would be relatively better than the original group of which they had become the survivors.

By increasing one or more of the three constants in Makeham's formula, Professor Whitney obtains seven types of sub-standard lives, which I take the liberty of grouping as follows:

Case II: Constant addition to the force of mortality.

Case III: Constant percentage addition to that portion of the force of mortality which increases with age. This is equal to a constant addition to the age.

Case IV: Combination of I and II, producing an addition to the force of mortality, increasing less rapidly than that resulting from a constant addition to the age.

Cases V to VIII: Combinations, each of which in practice would produce an addition to the force of mortality, increasing more rapidly than that resulting from a constant addition to the age.

In practice, Case II may fairly represent some occupational hazards, though even in such cases it will, theoretically, be unsuitable at the extreme old ages, a matter, however, of little consequence.

Cases III and IV were found by Mr. A. T. Winter (*J. I. A.*, XLIII, 370) as approximately representing, respectively, the rates of mortality among native lives in India and among Europeans resident in India.

Cases V to VIII appear to me to be such as will never occur in practice, and I doubt whether Case III occurs, except in connection with natives of tropical climates. For ordinary domestic business I think Case III—addition to age—an unsatisfactory method of rating-up, because the excess mortality is wrongly distributed, far too little being allocated in the early years and far too much in later years. If this contention be correct those cases, V to VIII, which locate a still greater proportion of the excess mortality in the later years, are still more decidedly inadmissible.

Using Hunter's Makehamized American Table, an addition of ten years to the age gives the following results:

At age 20 the excess mortality is	7 per cent.
At age 30 the excess mortality is	18 per cent.
At age 40 the excess mortality is	43 per cent.
At age 50 the excess mortality is	86 per cent.
At age 60 the excess mortality is	130 per cent.
At age 70 the excess mortality is	151 per cent.

I think it will be generally admitted that even those impairments whose effect is supposed to be long postponed will not show an excess mortality curve similar to that corresponding to the above

table. Some evidence on this point may be obtained from a study of the Specialized Mortality Investigation. Groups 49, 50, 56, 59, 60, 61, 63 and 64 showed badly in the later years of insurance; but the results in none of these groups indicated a rate of growth of excess mortality at all comparable with the above table, with the possible exception of group 49, consisting of persons with personal history of gout. The data in this group were too small to make the departure from the general results significant.

The method of adding a number of years to the age arose in Great Britain at a time when ordinary life policies formed the only considerable section of the business. It was a rough means of assessing the necessary extra premiums, and gave sufficiently serviceable practical results. The application of the theory to limited-payment policies, and to endowment insurances, gave extra premiums very much too small, supposing the addition correct for ordinary life policies. For instance, if a life aged 30 were rated up seven years on account of consumptive family history, most of the extra premium would arise from the assumed excess mortality after age 50, while, in reality, practically the whole of the excess mortality would occur before age 50. If a sufficient extra premium were charged, the difference between assumption and fact would not be very important, but the same assumption under a 20-year endowment insurance would largely underestimate the excess mortality of the first twenty years, and there would be no subsequent period of overestimated excess mortality to act as a counterbalance.

The constant addition to the force of mortality of Case II, and the increasing addition of Case IV, where the increase is at a slower rate than that arising from a constant addition to the age, are suited to certain sub-standard cases, but are, I think, not very common.

The typical case of sub-standard lives is, I believe, represented by an excess mortality rising to a maximum within a relatively short period after entry and subsequently gradually declining to comparative insignificance at the very old ages. It is thus a factor, both of age and duration, of insurance, and is, therefore, beyond the range of Professor Whitney's theory in its present stage of development. In order to meet actual conditions I think it will be necessary for any such theory to take account of selection.

I am not prepared to say that a development of Professor Whitney's theory may not be made practically useful. If so, the constants B and c will usually require to be adjusted in opposite directions, i. e., if one is increased the other must be *reduced*. Professor Whitney apparently does not contemplate a reduction of any of the constants entering into the force of mortality for standard lives, but this appears to me to be necessary to render the suggestion at all of practical value. The problem of adjustment to allow for selection is one that will probably call for considerable skill in its solution, and possibly Professor Whitney will feel disposed to deal with this matter later on.

MR. MEAD:

The publication of the Medico-Actuarial Investigation will widen extensively our underwriting knowledge and will enable us to assess the value of many classes of life risks with considerable accuracy. A result of this wider knowledge and of the greater confidence arising from it will doubtless be to gradually make more general the practice of accepting non-standard risks at appropriate ratings. Therefore, Professor Whitney's paper at this time is doubly welcome as a most important contribution to our knowledge of the theory and principles underlying the methods of sub-standard rating. Indeed, his demonstration is perfectly general and includes in a most admirable manner all the principal sub-standard methods now in vogue which require extra rating, and he, as well, indicates what may be termed a synthetic method of effectively building up the force of mortality from the non-standard elements that enter into it.

Professor Whitney employs, as a basis of his method, the convenient expression of the law of mortality as developed by Makeham, and proceeds to show how the constants may be modified in accordance with our underwriting knowledge so as to secure appropriate ratings.

Even a slight increase in the value of the γ constant or $\log_e c$, as it is commonly known, results in a large increase in the premium rate. For instance, at age 35 a 25 per cent. increase in the value of γ or $\log_e c$ results in an extra premium somewhat larger than that resulting from 100 per cent. increase in the rate of mortality, whereas a like increase in γ at age 45 results in a premium rate 25 per cent. higher than that arising from 100 per cent. increase in the rate of mortality. As Professor Whitney points out, an increase in γ is cumulative with age and applies principally to dissolute or intemperate lives or those subject to highly degenerative pathological conditions; in other words, to lives which should be rejected by the insurance companies. Therefore, for most practical cases the γ' of non-standard cases may be considered equal to the γ of standard lives, and we can secure the requisite increase in ratings by the proper modification of the α and β constants.

Professor Whitney's Case I is that of the standard life, and Cases II, III and IV are the only ones which do not involve a modification of the γ constant, and the particular point to which I desire to call attention is that these cases are already employed by life insurance companies in connection with their non-standard ratings.

Case II involves an increase in the α constant only, and this is equivalent to a constant increase in the rate of mortality, and results practically in a constant extra premium for all ages. This method of rating is especially applicable to occupational risks in which the extra hazard is limited for the most part to accidental

causes. We are familiar with the fact that a constant addition to the rate of mortality may be provided for by an addition to the rate of interest and the calculations thus considerably simplified. This is the result at which Professor Whitney arrives by his new general demonstration.

In Case III the β constant only is modified, that is, increased, and as Professor Whitney points out, this is equivalent to rating up the age.

In Case IV the α and β constants are both increased by the same factor, and while it may not be at once apparent, this is equivalent to assuming an increase in the rate of mortality by a constant extra percentage, which is a frequent method of assessment for the extra risk. In fact, Mr. Hunter has already indicated to us upon several occasions how his company, in actual practice, makes allowance for the increased frequencies of death by the additive process as indicated by Professor Whitney on page 294 of his paper. As has just been stated, Case IV is equivalent to assuming a constant extra percentage of mortality, and this extra percentage is usually obtained by the additive process. Mr. Hunter's company translates this into an advance in age, while one other prominent company publishes special tables of rates resulting from certain assumed percentages of extra mortality according to the American Experience Table. As this method of assuming a constant percentage of extra mortality is a very important and convenient one, it might be interesting to demonstrate mathematically that Professor Whitney's Case IV is the same as this method, and by so doing obtain by an alternative method formulas equivalent to Professor Whitney's final formulas, (11) and (12), for obtaining the increase in rate of interest and in age when γ' is equal to γ. (*Cf.* paper by H. E. W. Lutt on Extra Premiums, *J. I. A.*, XLI, 487). Since this is true, it follows that the premium rates according to an assumed extra percentage of mortality may be accurately obtained by employing a certain increase in age combined with an increase in the rate of interest.

Let

$$\mu_{x'} = (1 + m)\mu_x.$$

Employing the notation of Professor Whitney as given on page 285,

$$\mu_x = \alpha + \beta_0 e^{\gamma x},$$

and

$$\mu'_x = (1 + m)(\alpha + \beta_0 e^{\gamma x}),$$

where x is the standard life and x' the non-standard life subject to a constant extra percentage of mortality equal to m.

$$\therefore d \log l'_x = -\int [(1 + m)\alpha + (1 + m)\beta_0 e^{\gamma x}]\, dx,$$

$$\log l'_x = -(1 + m)\alpha x - \frac{(1 + m)\beta_0 e^{\gamma x}}{\gamma} + K,$$

whence

$$l_{x'} = + Ke^{-\alpha x(1+m)} e^{\frac{\beta_0}{\gamma}(1+m)e^{\gamma x}},$$

$$l'_{x'+n} = Ke^{-\alpha(x+n)(1+m)} e^{-\frac{\beta_0}{\gamma}(1+m)\, e^{\gamma(x+n)}}.$$

$${}_np_{x'} = e^{-\alpha n(1+m)} e^{-\frac{\beta_0}{\gamma}(1+m)e^{\gamma x}(e^{\gamma n}-1)}.$$

Multiplying both sides by v^n,

$$v^n_n p_{x'} = v^n e^{-\alpha n(1+m)} e^{-\frac{\beta_0}{\gamma}(1+m)e^{\gamma x}(e^{\gamma n}-1)}.$$

If now we find an age w so that

$$(1 + m)e^{\gamma x} = e^{\gamma w},$$

we get

$$v^n\, {}_np_{x'} = (ve^{-\alpha m})^n e^{-\alpha n} e^{-\frac{\beta_0}{\gamma} e^{\gamma w}(e^{\gamma n}-1)}$$

$$= (ve^{-\alpha m})^n\, {}_np_w \text{ or } v'^n\, {}_np_w.$$

We now solve for w and i' from the relations,

$$(1 + m)e^{\gamma x} = e^{\gamma w} \quad \text{and} \quad (ve^{-\alpha m})^n = v'^n,$$

$$\gamma w \log e = \gamma x \log e + \log (1 + m),$$

$$(w - x)\gamma \log e = \log (1 + m).$$

$$w - x = \frac{\log (1 + m)}{\gamma \log e},$$

which is equivalent to Professor Whitney's formula (12) when $\gamma' = \gamma$ and β'_0 is equal to β_0 times a constant.

Again

$$ve^{-\alpha m} = v',$$

$$\frac{e^{-\alpha m}}{1 + i} = \frac{1}{1 + i'},$$

$$\log_e (1 + i') - \alpha m = \log_e (1 + i),$$

$$\delta' = m\alpha + \delta,$$

which is equivalent to Professor Whitney's formula when α' is equal to α times a constant.

Professor Whitney's Case IV, where both the α and β constants are multiplied by two, is the same as the assumption of 200 per cent. mortality and results in the increase of the rate of interest from 3 per cent. to 3.79 per cent. and an increase in the age of

6.57 years. 25 per cent. extra mortality requires an increase in the rate of interest from 3 per cent. to 3.2 per cent. and an increase in age of 2.12 years, whereas 50 per cent. extra mortality results in a rate of interest of 3.39 per cent. and a rating up of 3.84 years.

ORAL DISCUSSION.

MR. HENDERSON: The method proposed by Mr. Whitney for the treating of sub-standard lives is certainly suggestive and ingenious, and it amounts in effect to considering the impairment as a combination of three different types of impairment. First, an increase in the chances of death, which is constant. It may be due to some permanent increased hazard or environment or to some other factor. Secondly, an absolute permanent deterioration beyond the average for the present age of the person in question; and thirdly, an increase in the rate of deterioration of the life insured, the effect being shown in the three constants of the Makeham formula, which he adopts as the basis, but it is not necessary to state his assumptions in connection with Makeham's law, as the whole of his mathematical analysis can be carried out on the plain assumption that the force of mortality is $k + n\mu_{n x+a}$. It can then be shown that any premiums for this impaired life can be calculated from the premiums of normal lives by using a factor, except that with a constant added to the force of mortality the rate of interest has to be raised. It seems to me that, ingenious as it is, it will not cover all the cases that actually arise in practice, nor even most of them. My belief is, that a different system would approximate closer to a large percentage of the cases, and that system is, to suppose that a group of impaired lives with a given impairment can be analyzed into two subgroups; in one of which the mortality would be the same as that at an age rated up a given number of years, say A years; in the other of which the mortality would be equal to that at another number of years rated up, say B years; and the percentage in these two groups could be determined so as to conform to an additional condition giving three constants to adapt the resulting mortality table to any observed series of mortality percentages. On that assumption the annuity values are easily calculated because they are simply the proportion of the annuity values of the two ages, remembering this in connection with the reserves, that when we are calculating the annuity values ten years hence, the proportion of the two classes will not be the same as now, because in the more highly rated up group there will be a smaller percentage of survivors than on the less highly rated up group; and this gives a considerable degree of flexibility in the application of the resulting table to various kinds of impairment. We can start with a very high initial impairment and run it down to a very low one by properly adjusting the two periods of rating and the proportion of the two groups.

Book Notices.

Medico-Actuarial Mortality Investigation. Volume II. Compiled and Published by the Association of Life Insurance Medical Directors and the Actuarial Society of America, New York, 1913. Pp. 139.

This volume constitutes a continuance of the report of the Joint Committee in charge of the Medico-Actuarial Mortality Investigation. It covers an investigation of the influence of build on mortality among men and women, an analysis of the causes of death among men and women, the mortality among women, and the mortality among certain races.

The investigation on the question of the influence of build on mortality among men was directed in part to an attempt to ascertain the departure, if any, of the weight which furnished the lowest rate of mortality from the average weight for the height and age of the applicant. Incidentally, of course, the results produced show the mortality experienced for various degrees of overweight and underweight at the different ages at entry. In studying the results of this investigation it should be remembered that all of the lives which entered into it were accepted as standard lives by the various companies, and, therefore, those whose weight departed materially from the accepted standard at the time when the policy was issued were probably subjected to a more rigid criticism in other respects before being accepted than were those whose weight conformed to the standard. It is probable, therefore, that the results of the experience somewhat understate the extra mortality which would result, other things being equal, from a departure from the standard weight. In making this investigation the various weights were combined in groups of five pounds, the central weight of each group being a multiple of five pounds and the group to which the average weight for the height and age of the applicant would have been assigned was taken to represent the average weight. Different combinations were then made of these groups in order to secure a sufficient volume upon which to base the experience. The following condensed table shows the percentage of actual to ex-

pected claims for ages at entry up to 34 inclusive, for ages 35 to 44 and for ages 45 and over:

Variation from Average Weight in Pounds.	Ages to 34, Per Cent.	Ages 35 to 44, Per Cent.	Ages 45 and over, Per Cent.
−25 to −45	114	100	88
−15 " −20	110	96	93
− 5 " −10	103	88	93
Average	99	97	93
5 to 10	96	97	102
15 " 20	90	105	117
25 " 45	115	136	124
50 " 80	126	165	148

It will be noticed that for the younger ages the minimum percentage occurs in the group from 15 to 20 pounds overweight, but that beyond this the rate of mortality increases rapidly, so that the group for 25 to 45 pounds overweight shows substantially the same extra mortality as the group from 25 to 45 pounds underweight. As the age advances, however, the point of minimum mortality appears to fall below the average weight to such an extent as to indicate, when combined with the results at the younger ages, that a standard weight for each height regardless of the age would serve better as a basis of comparison than the average table based upon both height and age which is at present used. Another point which is brought out in this table is that the percentages of extra mortality due to overweight do not appear to be so high for advanced ages at entry as for the central ages, at least where the overweight is 25 pounds or more. Possibly this may be due to the fact that the condition of the applicant in this respect has become more settled at these ages and the medical examiners are able to make a better selection among the risks presented. The investigation of the effect of build on mortality among women produced similar results. The volume of the experience was, however, not considerable and appeared to indicate, so far as it went, that the effect of underweight or overweight, particularly at the younger ages, is less than among men though exhibiting the same general tendencies.

The analysis of the causes of death among men was prepared for the purpose of comparison with similar tables based on the lives of those engaged in hazardous occupations or showing medical impairment. This analysis was based on 17,792 deaths recorded in the experience. The tables show the percentage from each cause to the

total number of deaths from all causes, and also the ratio of the deaths from each cause per 10,000 exposed to risk. These percentages are shown for ages at entry 15 to 29, 30 to 44 and 45 and over, and indicate clearly typhoid fever and tuberculosis of the lungs as especially diseases of the younger ages. In the case of appendicitis, accident and suicide, the percentage of deaths from these causes to the total number of deaths is greater at the younger ages, but the percentage of the exposed to risk is less. The diseases which show the greatest proportionate mortality at the advanced ages are cerebral hemorrhage and apoplexy, and cancer. An analysis of the causes of death of those underweight 25 pounds and more and those overweight fifty pounds and more, shows at all ages that the percentage of deaths from tuberculosis among the heavyweights is markedly less than amongst standard lives; whereas, in the case of such diseases as nephritis, organic diseases of the heart and apoplexy, the reverse is the case. Underweights show variations from the standard exactly opposite with respect to these diseases to those shown by the heavyweights, the mortality from tuberculosis being excessive and that from the other diseases being less, except that from organic diseases of the heart for young ages at entry, which is substantially the same as for standard lives. The analysis of the causes of death among women shows similar results regarding these diseases. Comparing the causes of death among women with those among men it is found that the death rate from typhoid fever, appendicitis and typhlitis, cirrhosis of the liver, suicide and accident is greater among men than among women, while there is a greater proportion from cancers and other malignant tumors among women than among men.

In investigating the mortality among women they were divided into four classes:

(*a*) Spinsters.
(*b*) Married women, beneficiary husband.
(*c*) Married women, beneficiary other than husband.
(*d*) Widows and divorced.

The following table shows a brief summary of the experience in these four classes:

	Number Entering.	Actual Deaths.	Expected Deaths.	Ratio of Actual to Expected Deaths.
Spinsters	149,519	3,557	4,412	81%
Married women, beneficiary husband	94,813	3,235	2,566	126
Married women, beneficiary other than husband	82,973	4,403	3,869	114
Widows and divorced	65,727	4,312	4,124	105
Total	393,032	15,507	14,971	104%

The general results of this experience indicate that the mortality among spinsters has been much more favorable than among men, that there has been no greater adverse selection than among men, and that, compared with men, the relative mortality decreases with advancing age at entry. On the other hand, in the two classes of married women the mortality has been distinctly higher than among men and consequently much higher than among spinsters, and, while marked selection against the companies is indicated by the experience in the early policy years, no marked difference was observed as between the two classes of married women. Widows show a mortality between that of spinsters and that of married women, with apparently no more selection against the companies than among men.

The only race of which any considerable volume of experience could be secured was the race of negroes which was separated into two classes, (*a*) colored ministers, teachers and other professional men, (*b*) all other colored men.

The following table shows a synopsis of the experience among negroes:

	Number Entering.	Actual Deaths.	Expected Deaths.	Ratio of Actual to Expected Deaths.
Ministers, teachers and other professional men	4,503	202	147.4	137%
All other colored men	17,296	792	540.0	147

Abstract from the Minutes of the Annual Meeting of the Actuarial Society of America, held in New York City on Thursday and Friday, May 15 and 16, 1913.

Hotel Astor, New York City,
May 15, 1913.

FIRST DAY.

The meeting was called to order by the President, Mr. Macdonald, at 11 o'clock A. M. The following members were present:

FELLOWS.

Alsop,
Blehl,
Brinkerhoff,
Carpenter,
Craig, J. D.,
Craig, J. M.,
Dow,
Fackler, D. P.,
Fackler, E. B.,
Fitzgerald,
Flynn,
Gaylord,
Gore,
Grow,
Henderson,
Hunter, A.,
Hunter, R. G.,
Huntington,
Hutcheson,
Ireland,
Jackson,
Joffe,
Johnston,
Kilgour,
Laird,
Langstaff,
Linzmeyer,
Little,
Lunger,
Macaulay,
Macdonald,
Maclean,
Marshall,
Mead,
Moir,
Morris,
Mowbray,
Murphy,
Papps,
Perrin,
Reiter,
Rhodes,
Richter,
Salter,
Sheppard,
Stilwell,
Strong,
Thompson,
Watt,
Welch,
Wells,
Wood, A. B.,
Wood, W. A. P.,
Woodword,
Woodward,
Young.

ASSOCIATES.

Allison,
Allstrom,
Angell,
Brough,
Brown,
Carrington,
Cathles,
Coburn,
Davenport, J. S.,
Fairlie,
Forster,
Gould,
Graham,
Hemsing,
Homans,
Howe,
Hughes,
Kaufman,
Kime,
King,
Linton,

MACFARLANE,	PHILLIPS,	THOMAS,
MACLEAN,	RICE,	VINEBERG,
MARSHALL,	RYAN,	WALKER,
MOODIE,	SMITH,	WASHBURNE,
MOORE,	SPEERS,	WILLIAMS.
OLIFIERS,	STRONG,	

After the calling of the roll the President made his address to the Society.

The minutes of the Semi-Annual Meeting, held in Toronto, October 17 and 18, 1912, were approved as printed in the *Transactions*.

The Secretary recorded the proceedings of the Council since the date of the last meeting.

The report of the Treasurer was then read, and, on motion, approved.

The Chairman of the Central Bureau of the Medico-Actuarial Mortality Investigation presented a report of the progress of that investigation.

The Society then proceeded to the election of Officers and Members of Council with the following result:

President: William C. Macdonald.

Vice-Presidents: James M. Craig, Arthur Hunter.

Secretary: Robert Henderson.

Treasurer: David G. Alsop.

Editor of the Transactions: Wendell M. Strong.

Members of Council for three years: Edward E. Rhodes, Morris W. Torrey, Joseph H. Woodward.

An invitation was presented from the members of the Society residing in Boston to hold the fall meeting in that place. On motion, the invitation was accepted.

The Society adjourned at one o'clock.

AFTERNOON SESSION.

After the Society had reassembled at 2:30 P. M., the reading of papers prepared for this meeting was proceeded with. After the reading of the papers was completed the Society took up the discussion of the papers presented at the meeting in October, 1912.

At five o'clock the Society adjourned to meet at dinner at seven o'clock and to reassemble for business at the following morning at ten o'clock.

SECOND DAY.

The meeting was called to order by the President.

The discussion of papers read at the meeting October, 1912, was resumed and completed.

On motion of Mr. Hunter, the following resolution was adopted:

WHEREAS, there has been offered in the Senate of the United States, by Hon. Morris Sheppard, and referred to the Committee on Education and Labor, the following resolution:

> "*Resolved,* That the Secretary of Labor be, and he is hereby, directed to investigate and report, as far as it is practicable, upon the mortality and the disability by accident or by disease incident to or resulting from the various occupations in which the wage earners of the United States are engaged."

WHEREAS, we learn that the Secretary of Labor has expressed to such Committee his willingness to undertake this task and his belief that such an investigation is most desirable; and

WHEREAS, a large amount of data bearing upon mortality in the various occupations is available, in the census returns and in the returns of fraternal societies, for the purposes of such an investigation,

NOW, THEREFORE, the Actuarial Society of America, having largely completed a similar joint investigation with the Association of Medical Directors and recognizing the great importance and value of an investigation such as is here proposed, adopts the following resolution:

Resolved, that we favor the investigation of mortality and disability proposed in the resolution offered in the United States Senate by Hon. Morris Sheppard, Senator from Texas, as likely, if carried out with thoroughness and care, to furnish much information of the highest value and utility; and

Resolved, that the Secretary of the Society be directed to send copies of these Resolutions to Hon. Hoke Smith, the Chairman, Committee on Education and Labor, United States Senate and to Senator Sheppard.

At one o'clock the Society finally adjourned.

LEON MARIE.

By the death of M. Leon Marie, Secretary General of the *Institut des Actuaries Francais,* a notable figure has been removed from the Actuarial profession. He was born at Paris August 12, 1854, and entered the actuarial service of the *Phenix* Life Insurance Co. in 1879 and at the time of his death he was Assistant Manager of that company. In addition to his duties in connection with that company, he served from 1896 to 1900 as professor in the *Ecole des Hautes Etudes Commerciales,* having previously served as examiner from 1881. In 1890 he published his "Traite des operations financieres" and in 1893, as reporter for a special commission, he wrote the "Rapport sur la Comptabilite statistique et financiere des Societes de Secours mutuels," which established his reputation as an authority on the subject. He took a prominent part in 1890 in the organization of the *Institut des Actuaires Francais* and served as its General Secretary from that date until his death. In that capacity he contributed largely to its success. He was one of the Committee of four actuaries who constructed the A.F. and R.F. Tables of Mortality which have since been the standard tables of mortality in France. He served as a member of the Advisory Commission whose labors resulted in the French law of March 17, 1905, regarding life insurance, and he was appointed on the Advisory Committee on Life Insurance of the French Insurance Department organized under that law. Besides his membership in the *Institut des Actuaires Francais* and in the Actuarial Society of America, he was a corresponding member of the Institute of Actuaries of Great Britain and a member of the *Association des Actuaires Belges* and was one of the representatives of France on the Permanent Committee of the International Congresses of Actuaries.

ERNST WILLEM SCOTT.

Ernst Willem Scott, for many years Managing Director, Algemeene Maatschappij van Levensverzekering en Lijfrente, died on January 22, 1913. He was born at Rotterdam, Holland, on May 16, 1836, and was one of the early members of the Actuarial Society of America, his membership dating back to April 28, 1893. He was also an Associate of the Institute of Actuaries of Great Britain from 1873. His connection with the Algemeene Maatschappij as Managing Director dated from its foundation in 1880.

Examinations of the Society

Held on May 21 and 22, 1913.

Examination Committee.

General Chairman, . . William Young.
Vice-General Chairman, Arthur B. Wood.

For Fellowship.
Joseph H. Woodward (Chairman),
Albert G. Portch,
Herbert N. Sheppard.

For Associateship, Section B.
John S. Thompson (Chairman),
Louis Linzmeyer,
M. S. Hallman.

For Associateship, Section A.

Part I.
Edmund E. Cammack (Chairman),
Albert H. Mowbray,
John M. Laird.

Part II.
Lorne K. File (Chairman),
William A. Watt,
Ray D. Murphy.

EXAMINATION FOR ENROLMENT AS ASSOCIATE.

Section A—Part I.

1. (*a*) State and prove the rule for converting a recurring decimal into a common fraction.

(*b*) A corporation redeems \$2,100,000 of 5% bonds at 105 under an option agreement, raising all funds required for the purpose by an issue of 4% bonds which it markets at 98, brokerage being at ⅛% on the market price and other expenses ¼% on the par value of the new issue. What is the resultant change in the annual interest charge?

2. (*a*) If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ show that

$$\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}+\frac{z^3+c^3}{z^2+c^2}=\frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}$$

(*b*) A cask P is filled with 50 gallons of water, and a cask Q with 40 gallons of brandy; x gallons are drawn from each cask, mixed and replaced; and the same operation is repeated. Find x

if there are 8⅞ gallons of brandy in P after the second replacement.

3. Solve

(*a*) $x^2-xy+y^2=7$
$x^4+x^2y^2+y^4=133$

(*b*) $x+y+z=13$
$x^2+y^2+z^2=65$
$xy=10$

(*c*) $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\sqrt{\frac{7}{xy}}+1$
$\sqrt{x^3y}+\sqrt{y^3x}=78$

4. (*a*) Show that for all real values of x the expression ax^2+bx+c has the same sign as "a" except where the roots of the equation $ax^2+bx+c=0$ are real and unequal, and "x" has a value lying between them.

(*b*) If x be real, prove that $\frac{2x-7}{2x^2-2x-5}$ can have no real value between 1 and 1/11.

5. (*a*) What are the meanings of a^0, a^{-m}, $a^{1/m}$ where m is a positive integer. Show how these symbols come to have such meanings assigned to them.

(*b*) Show that the square root of $p-\sqrt{q}$ where $\sqrt{q}$ is a surd can be expressed as a simple surd provided p is positive and p^2-q is a perfect positive square.

(*c*) If $m=a^x$
$n=a^y$
$m^yn^x=a^{z^2}$ show that $xyz=1$.

6. (*a*) Find the total number of ways in which "n" things may be arranged among themselves, taken all at a time when "p" of the things are alike of one kind, "q" alike of another kind, and "r" alike of a third kind, and the rest all different.

(*b*) In how many ways can 30 silver dollars be given to 8 persons, without giving less than 2 to any person.

7. (*a*) Prove by induction the Binomial Theorem for a positive integral index.

(*b*) By means of the Binomial Theorem find the value of $\sqrt[3]{124}$ to five decimal places.

(*c*) Find the co-efficient of x^{15} in the expansion of $\left\{x^2+\frac{4c}{x}\right\}^{13}$

8. (*a*) Explain what is meant by Convergency and Divergency of a series.

(*b*) Find whether the following series is convergent or divergent:

$$\frac{x}{1+x^3}+\frac{x^2}{1+x^4}+\frac{x^3}{1+x^5}+$$

and show that $1+1/2^x+2/3^x+3/4^x+$ is convergent if $x>2$ and divergent if $x<$ or $=2$

9. Sum to infinity

(*a*) $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5.}{4.8.12}+$

(*b*) $\frac{3}{1.2.4.}+\frac{4}{2.3.5.}+\frac{5}{3.4.6.}+$

10. (*a*) Assuming the Exponential Theorem, deduce the Logarithmic Series

(*b*) If x lies between 1 and 2, show that

$$x-1>\log x>1-1/x$$

(*c*) Given log 2=.3010300
log 3=.4771213 Show $\left(\frac{81}{80}\right)^{1000}$ exceeds 100,000.

11. (*a*) Explain the meaning of the word "Probability" in Mathematics.

(*b*) If $p_1, p_2, p_3, \ldots . p_n$ be the respective probabilities of the happening of "n" independent events in a certain year, what will be the the chance that in that year

(1) All the events will happen.
(2) None will happen.
(3) One particular event will happen, and the others fail.
(4) At least one will happen.
(5) Exactly one will happen.
(6) At least one will fail.

12. (*a*) If the probability of the happening of an event in any trial is p, and the probability of its failing is q, find the most probable number of happenings in n trials.

(*b*) A bag contains 5 white and 3 black balls, and four are successively drawn out and not replaced. What is the chance that they are alternately of different colors?

(*c*) A man tosses 10 pennies; removes all that fall head up; tosses the remainder and again removes all that fall head up; how many times ought he to be allowed to repeat this operation in order that there may be an even chance that before he is done all the pennies have been removed?

(Answer only two of the questions 13, 14, 15).

13. (*a*) Prove that in a right-angled triangle the midpoint of the hypotenuse is equidistant from the three vertices.

(*b*) Show that the opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.

14. (*a*) Divide a straight line internally into two parts so that the rectangle contained by the whole and one part shall be equal to the square on the other.

(*b*) Show that the midpoints of a set of chords of a circle which all pass through the same point lie on another circle.

15. Find a point o within a triangle ABC, such that angle oAB=angle oBC=angle oCA.

SECTION A—PART II.

1. (*a*) Derive a formula for u_{x+mh} in terms of u_x and its differences. Outline the method of converting such a formula into terms of central differences and state central difference formula.

(*b*) Given $\mu_{3:5}$, $\mu_{3:10}$, $\mu_{3:15}$
$\mu_{7:2}$, $\mu_{7:7}$, $\mu_{7:11}$
$\mu_{12:4}$, $\mu_{12:11}$, $\mu_{12:12}$

outline how you would obtain the value of $\mu_{6:9}$, giving the formula or formulas you would use.

2. (*a*) Prove that $\triangle^n(u_x v_x) = v_x \triangle^n u_x + n \triangle v_x \triangle^{n-1} u_{x+h} + \frac{n(n-1)}{1.2} \triangle^2 v_x \triangle^{n-2} u_{x+h} + \cdots$

(*b*) What formula could be used without computing these several differences if the values of u_x and v_x were known for the values of x from x to $x+nh$ inclusive?

3. (*a*) Prove that $f(0)+f(1)+\ldots+f(n-1)$

$$= nf(0) + \frac{n(n-1)}{1.2} \triangle f(0) + \frac{n(n-1)(n-2)}{1.2.3} \triangle^2 f(0) + \cdots$$

(*b*) Sum to 15 terms the series

$$1+16+81+256+625+\ldots\ldots$$

and state whether the result is exact or approximate and why.

4. (*a*) Give a verbal interpretation of the equation

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

(*b*) Derive a formula for the present value of an annuity for 20 years, first payment 10 years hence, payments to be made semi-annually; interest at 4 per cent. compounded quarterly.

5. Holders of Railroad 5's maturing 15 years hence at 110, and now quoted at 116 are offered in exchange, on the basis of market values, 4% bonds quoted at 90 and redeemable 25 years hence at par. Interest in each case is payable yearly. As an investment would you advise the exchange?

Given. $v^{15}=.55526$, $v^{25}=.37512$ at 4%.
$v^{15}=.48102$, $v^{25}=.29530$ at 5%.

6. A sells B a house for \$10,000 to be paid for in ten equal annual instalments, beginning one year from the present date. These instalments are to be of such an amount as to yield 7% on the entire \$10,000 until payments are completed. On the assumption that A can invest the annual principal repayments at 5%, find the amount of each annual instalment.

Given $(1.05)^{10}=1.6289$.

7. Draft a form of Cash Book, showing the columns usually included.

State how this book should be balanced. What is the test of its correctness?

8. Obtain the differential coefficient of y with respect to x in the following

(a) $y=\dfrac{x^3(x^2-a^2)^{\frac{1}{2}}}{x+a}$

(b) $(x^2+y^2)^2-a^2(x^2-y^2)=0$

(c) $y=\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$

9. Evaluate

(a) $\int\dfrac{x^4-5x^3+6x-4}{x^3-4x^2-5x}\,dx$

(b) $\int\dfrac{\sqrt{x^2+a^2}}{x^2}\,dx$

(c) $\int\dfrac{dx}{x\sqrt{1+x+x^2}}$

10. (a) Evaluate $\int_0^2\dfrac{dx}{\sqrt{2-x}}$

(b) Obtain the nth differential coefficient of y with respect to x in the function, $y=x^2e^{ax}$.

(c) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ where $x=\dfrac{1-t}{1+t}$ and $y=\dfrac{2t}{1+t}$

11. A loan of S is to be redeemed in t years with interest at the effective rate x, by equal instalments of principal and interest payable r times a year. Draw up a schedule showing how the successive payments are divided between interest and principal repayments.

Show the identity of this form of repayment with that in which the sinking fund instalments are invested independently of the loan.

12. A 20-year 6% bond with half yearly coupons and redeemable at par is bought at 134. Find the investment yield on the following suppositions:

(*a*) That the purchaser writes his investment down each half year by the excess interest in the coupon.

(*b*) That the purchaser sets up a sinking fund at 4% compounded half-yearly to replace the premium at which he bought the bond. Given v^{40}=.49960 at 1¾%.

v^{40}=.47566 at 1⅞%.

$s_{\overline{40}|}$=60.40198 at 2%.

13 (*a*) A 20-year bond of $1,000 is sold to net the purchaser 4½% on his investment. The bond bears interest at 4% payable half yearly for the first 10 years and 5% payable annually for the last 10 years. Give a numerical expression for its purchase price.

(*b*) Find the value of an annuity payable annually whose several payments are 1, 2, 3, 4, etc., when the annuity is to run

(i) for n years.

(ii) forever.

14. (*a*) State briefly but clearly in their proper order, the steps necessary to close the books and prepare the Balance Sheet of a wholesale trader and illustrate in skeleton his Profit and Loss Account and Balance Sheet with six entries in the former and four entries on each side of the latter.

(*b*) At the close of the financial year, rents $300 and interest on investments $500 have been earned but are not yet payable. How would you deal with these in the Profit and Loss account? Show by means of journal entry how you would bring these on your books.

SECTION B—PART I.

1. Define Central Death Rate; in what connection does the necessity for the use of such a death rate arise?

Prove $m_x = q_x + \frac{q_x^2}{2} + \frac{q_x^3}{4} + \ldots$.

What are the conditions under which $m_x = q_x$ approximately?

2. (*a*) Find the value of the probability that (x) will die first of three lives, (x), (y) and (z).

(*b*) If the probability that (x) will die before (z) is .1996, the probability that (x) will die before both (y) and (z) is .1610, the probability that (y) will die before (z) is .2990, and the probability that (y) will die before both (x) and (z) is .2602, find the following probabilities:

(i) that the survivor of (x) and (y) will die before (z);
(ii) that (x) will die before (z), (y) having died first.

3. (*a*) Find the probability that one at least of the three lives (x), (y) and (z) will fail between the end of the 10th and the beginning of the 15th year from now.

(*b*) The probability that two persons aged respectively 20 and 40 will not both be alive at the end of twenty years is .38823. Out of 96,223 persons alive at age 20, 6,358 die before attaining age 30. Find ${}_{30}|q_{30}$.

(*c*) Determine, correct to five decimals, the probability that out of eight persons, all aged x, at least three will die within a year, given that one per cent. of persons aged x, die before attaining age $x+1$.

4. (*a*) Find the probability that (x) will die in the tth year, during the lifetime of (y) and having survived (z) by at least n years.

(*b*) Find, in a form suitable to select tables, the probability that (x) will be alive at the end of the tth year succeeding that in which (y) dies.

5. (*a*) Show that in a stationary population the average present age at any time is equal to the average expectation of life at age 0 and upwards.

(*b*) In a stationary community there are annually 209 deaths per ten thousand inhabitants. What is the average age at death?

6. Given a complete table of p_x applicable to a given community with a stationary population N, show how to find the

number dying annually and the number living at each age and the number of births in a year.

7. A joint 20-payment life policy provides the following benefits:

(*a*) An annuity-certain of a unit for n years, the first payment to fall due at the death of either of the assured.

(*b*) A deferred annuity of a unit on the life of the survivor, the first payment to fall due n years after the death of the other life.

Deduce the formula for the annual premium for such a policy.

8. (*a*) A, B, C, D and E are entitled to share equally in the income of an estate, the income each year being distributed among the survivors, the last survivor receiving the entire income for the rest of his life. Find an expression in single and joint life annuities for A's interest.

(*b*) Prove the formula, $a_{\overline{abc}:\overline{xyz}} = a_{\overline{abc}} + a_{\overline{xyz}} - a_{\overline{abcxyz}}$.

9. A company is asked to issue an Ordinary Life Policy on the life of a man age 40 with the proviso that if his wife age 37 dies first the company is to pay 10% of the face of the policy, the insured thereafter still having the insurance protection of the remaining 90%, the premium then being that charged for a regular Ordinary Life Policy for the reduced amount. How would you arrive at the proper premium to be charged upon the issue of the policy? Would it be necessary for the wife to be examined?

10. Give in simplest form the formula for the annual premium for the first 10-year period for a policy providing for the payment of $1,000 in case of death during the first ten years, the insurance increasing $100 per year during the second ten years and decreasing $100 per year during the following fifteen years, then remaining constant for a period of five years and finally being reduced one-half. Premiums payable annually for a period of thirty years, the premiums for the 11th to 20th years being 50% less than during the first 10 years and for the 21st to 30th 50% less than during the 11th to 20th years.

11. (*a*) Prove algebraically and interpret verbally the following formulae:

(i) $a_x = \dfrac{v - A_x}{d}$

(ii) $A_x = \dfrac{a_\infty - a_x}{1 + a_\infty}$

(iii) $P_x = \dfrac{dA_x}{1 - A_x}$

(*b*) Prove that

$$(a_x - a_{xy}) : (A_y - A_{\overline{xy}}) :: (1 + i) : i$$

12. What is the nature of the error involved in the approximate formula for the value of a complete annuity payable yearly,

$$\overset{\circ}{a}_x = a_x + \tfrac{1}{2} A_x \, (1 + i)^{\frac{1}{2}} ?$$

Deduce a correct expression on the assumption of a uniform distribution of deaths and thence obtain an approximate value for the amount of error involved in the above formula.

13. What is the average amount to which forborne annuities–due at age x will accumulate

(1) If all members receive equally at death, irrespective of the amounts they may have paid in,

(2) If all annuity payments are thrown into a common fund from which the representatives of each one dying withdraws only the accumulations of his particular annuity?

Under what conditions do these values become identical?

14. Derive a formula for the annual premium for a reversionary annuity to (x) after the death of (y), the annuity to be apportionable, and to be payable m times a year, first payment at death of (y), assuming that a certain rate of mortality is experienced while the annuity is in reversion and a lower rate while the annuity is in possession. It is also provided that in case (x) predeceases (y), all premiums are to be returned.

15. Find an expression for an annuity payable during the joint lives of (x) and (y) and for n years longer should the survivor live so long.

16. Obtain the single premium for a temporary assurance for ten years, one-half being payable at the first death and the other half at the second death of three lives (x), (y) and (z). What is the objection to making the annual premium for this policy a level one payable during the continuance of the policy? How may this objection be met?

17. (a) Give a brief account of the methods you would adopt in the construction of tables of values of net single and annual premiums for deferred annuities, and state how you would verify your results.

(b) Show in detail how to obtain a table of annual premiums for Endowment Assurances maturing at age 60 or earlier death, without constructing the life table, given only a table of q_x.

18. Having computed tables of log D_x and log C_x for all ages for the same mortality table but at two different rates of interest, how would you verify the accuracy of the results? And, given tables of M_x and N_x at a single rate of interest, how would you verify the accuracy of both tables in a single operation?

19. (a) What reasons have been advanced to account for the failures of numerous life insurance companies about forty years ago?

(b) What are the chief causes of failure of assessment insurance companies?

(c) In what year was the first valuation of policies made by the Insurance Department of the State of New York, and what was the basis of such valuation?

SECTION B—PART II.

1. (a) State Gompertz's and Makeham's laws of mortality respectively, demonstrating the forms assumed in each case by the functions μ_x and l_x.

Assuming Makeham's law to hold, find the differential coefficient of ${}_tp_x$,

(i) with regard to t.

(ii) with regard to x.

(*b*) Explain the principle of Uniform Seniority. To what mortality tables does it apply?

2. Employ the differential calculus to determine under what conditions the function $(1+i)P_x$ is diminished by increasing i.

3. State what $\int_0^\infty v^t \, {}_tp_{xyz} \, \mu_{z+t} \, \bar{a}_{x+t} \, dt$ represents and find its differential coefficient with respect to x.

4. (*a*) Evaluate (i) $\int_0^1 {}_tp_x \, \mu_{x+t} \, dt$, and (ii) $\int_0^\infty {}_tp_x \, \mu_{x+t} \, dt$ and give verbal interpretation of the result in each case.

(*b*) Interpret and obtain integral expressions for

$$\bar{A}^2_{xyz} \quad ; \quad \bar{A}^{1:3}_{x:yz} \quad ; \quad \bar{A}^{\quad\quad z}_{\underline{xy}:\underline{\quad}}{}_3 \, .$$

5. Express each of the following in the form of a definite integral,—

$$L_x \quad , \quad |_n g_{xy} \quad , \quad \mathring{e}_{xyz} \, .$$

In the case of the last mentioned, deduce the approximate correction to be added to e_{xyz} in order to obtain the complete expectation and find the numerical value of such correction in the case of three lives of equal age having given that the force of mortality for a single life at that age is one per cent.

6. Show that ${}_nV^{(m)}_x - {}_nV_x = \frac{m-1}{2m} \, P^{(m)}_x . \; {}_nV_x$ and give a verbal explanation.

7. Derive Hardy's formula

$$\int_0^\omega u_x dx = n \left\{ .28u_0 + 1.62u_n + 2.2u_{3n} + 1.62u_{5n} + .56u_{6n} + 1.62u_{7n} \right\}$$

and show how it may be applied to find the value of $\bar{a}_{yz}{}^{1}\big|_x$ explaining clearly how n is determined.

8. (*a*) Give five formulas for ${}_nV_{x\bar{1}}$ and prove algebraically that the first is the exact equivalent of the other four.

(*b*) Prove $(P_x+d)\,(1+a_{x+n})+{}_nV_x=1$.

Interpret the equality and hence state what the relation becomes in the case of a limited-payment life policy, (i) during the premium-paying period and (ii) after the policy is paid up.

9. An actuary in comparing a number of mortality tables finds that tables A and B show the same mortality at young ages, but at the higher ages an increasing difference in the rate of mortality. Tables B and C show the same mortality at the higher ages, but an increasing divergence in proceeding to the younger ages. Tables C and D show the same mortality at the middle ages, but the mortality of table C is higher at the younger ages and lower at the older ages. What general conclusions would you draw as to the reserves required as between tables A and B, B and C, C and D?

10. Having given terminal reserves on the Ordinary Life plan, the N_x and D_x columns, the net annual premium for an Ordinary Life policy and the net annual premium for a 20-Year Endowment insurance providing for return of all premiums paid in case of death after the 10th year, how would you calculate the reserve on the latter policy at the end of the 10th year? Give algebraic proof.

11. Theoretically $1-\frac{P_x}{P_{x+n}}$ represents the paid-up insurance that would be allowed at the end of the nth year, in exchange for an Ordinary Life policy.

Prove algebraically and interpret verbally.

Explain why this is not necessarily a correct formula in practice.

12. The premiums and guarantees for a 20-Payment Life policy are calculated on the American 3% basis. How would you value this policy at the end of the 10th year on the American 3½% basis?

13. Given two mortality tables for which reserves have not been calculated, how would you proceed to ascertain how the values, by the respective tables, of Endowment Assurances maturing at age 60 or at previous death, would compare? Prove your method fully.

14. In case of a mortality table which follows Makeham's Law, what is the effect on the policy value of an Ordinary Life policy, of a constant addition to the force of mortality?

15. Prove that the value at the end of the nth year of an Ordinary Life policy issued at age x may be denoted by

$$\frac{P_x - P^1_{x\,\overline{n}|}}{P_{x\,\overline{n}|}^{\;1}}.$$

Interpret this formula and show what it becomes in case of a Semi-Endowment policy.

16. What multiple of P_x must be added to the right-hand side of equation

$${}_nV_x = 1 - \frac{a_{x+n}}{a_x}$$

in order that it may be valid for other than integral values of n?

17. What clauses are usually or may be contained in Short Term policies which are not in life or endowment policies, and *vice versa*?

18. Enumerate the conditions and benefits which you would embody in a 20-Payment Life policy under present competitive conditions, and specify those which are absolutely essential as a mere contract of insurance, as distinguished from those which are the result of legal requirement.

(Candidate may answer Question No. 18 with reference to local legal requirements.)

19. Draft for an Ordinary Life Policy a non-forfeiture clause providing for cash, paid-up life insurance and paid-up extended term insurance, including provision for modifying values on account of dividend additions and outstanding indebtedness. How would the clause be changed to apply to an Endowment Assurance Policy?

20. What are the usual limitations embodied in a clause providing for waiver of premiums in event of total permanent disability?

EXAMINATION FOR ADMISSION AS FELLOW.

PART I.

*1-A. (*a*) Show that, by a mortality table which follows Makeham's law,

$$\bar{A}_x = -\log s\,\bar{a}_x + (\mu_x + \log s)\,\bar{a}_x'$$

where $\bar{a}_x'$ is calculated at a rate of interest "j" such that $\frac{1}{1+j} = \frac{c}{1+i}$.

(*b*) Using the calculus of finite differences show that

$$a_x^{(m)} = a_x + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x + \delta) - \text{etc.}$$

1-B. (*a*) Indicate what is known regarding the sources of the following mortality tables:

(1) Northampton;
(2) Carlisle;
(3) Farr's English Life, No. 3.

(*b*) The values of $1000q_x$ at ages 20, 40, 60 and 80 according to certain mortality tables are as follows:

TABLE

AGE	(1)	(2)	(3)	(4)	(5)
20	4.0	3.8	7.8	6.5	6.3
40	9.2	5.9	9.8	9.8	10.3
60	28.9	18.8	26.7	29.2	29.7
80	138.4	96.7	144.5	138.5	144.7

The tables represented are the American, McClintock's Female Annuitants, H^M, O^M, $O^{M(5)}$. Identify the table in each case, giving concisely your reasons.

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*Candidates who have passed the Associateship Examinations prior to the year 1910 will omit Questions 1-B and 2-B and will take Questions 1-A and 2-A.

Other candidates will omit Questions 1-A and 2-A and will take Questions 1-B and 2-B.

The remaining questions are for all candidates.

2-A. (*a*) Assuming the relation

$$\mathring{e}_x = \int_0^\infty {}_tp_x \, dt$$

prove that

$$\mu_x = \frac{1}{\mathring{e}_x}\left\{1 - \frac{1}{2}\left(\mathring{e}_{x-1} - \mathring{e}_{x+1}\right)\right\} \text{ approximately.}$$

(*b*) Show how an interpolation formula may be used to find the value of an annuity the first payment of which is to be made $\frac{1}{t}$ of a year hence.

2-B. (*a*) State what you know regarding the data upon which the Actuaries' or Combined Experience Table of Mortality is based, and how the mortality under this table compares with the American Experience.

(*b*) Summarize what is shown by the principal mortality tables compiled and published during the past one hundred years regarding the changes in the rates of mortality which have taken place during that period among (1) the general population; (2) insured lives.

3. (*a*) Given the census records of a large community taken decennially on March 31 and also the death records for calendar years, all being tabulated by quinquennial groups of ages, except 0—5, which are tabulated for each age, state how you would obtain the graduated p_x column for a table of mortality.

(*b*) Describe one method of graduating a mortality experience by Makeham's formula and discuss the value of this formula in dealing with a select experience.

4. (*a*) Describe in detail the methods used in constructing and graduating the $O^{M(5)}$ mortality table.

(*b*) A number of life insurance companies desire to combine their mortality experience in the form of a select table. What methods would you recommend for recording and tabulating the data in order to obtain the ungraduated values required?

5. A well-established mutual life insurance company, admitted to transact business in the State of New York, is revis-

ing its rate book. The officers decide that a consistent method should be followed in the calculation of premiums, surrender values and dividends.

(1) What table of mortality and rate of interest would you recommend for the calculation of premiums?

(2) How would you calculate the scale of surrender values?

(3) How would you provide for expenses and afterwards apportion them in calculating dividends? (Answer briefly.)

6. (*a*) What mortality tables, rates of interest and loading would you recommend for use in preparing office rates for (1) immediate annuities; (2) deferred annuities?

(*b*) How would you regard a suggestion that the office rates for immediate annuities should be varied from time to time according to the rate of interest which is being obtained on the current long term investments of the company?

(*c*) Under what circumstances, if at all, would you recommend the use of special office rates for annuities on impaired lives?

7. (*a*) Give and explain at least two formulae for the calculation of office premiums which specifically take into account the acquisition expenses of the business.

(*b*) Outline the considerations, both theoretical and practical, which should influence the grading of commissions.

8. (*a*) Define the expression "full level premium reserve system" as it appears in the "interrogatories regarding Gain and Loss Exhibit" and discuss the objections, if any, to this system and the extent to which they are overcome by the other methods of valuation mentioned in the interrogatories.

(*b*) What method would you recommend for valuing the outstanding policies of a life insurance company for the purpose of determining equitably the sum to be paid by the company in consideration for the reinsurance or transfer of its entire business?

9. (*a*) Suggest suitable forms for the principal cards and other office records which should be kept by a life insurance company, covering number of policies, amount of insurance, and

reserve, and which will enable the total policy reserve to be obtained immediately on closing business December 31.

(*b*) If you were employed to certify unconditionally to the policy reserve of a company with whose records you had no previous familiarity, what steps would you take to verify the valuation data?

10. State how you would proceed to ascertain the liabilities of a company as of December 31 under each of the following heads:

(1) Losses incurred, no proofs received;

(2) Losses and other policy claims resisted;

(3) Liability on policies canceled upon which a surrender value may be demanded;

(4) Premiums paid in advance;

(5) Bills and accounts due or accrued.

11. (*a*) Discuss the practical advantages and disadvantages of a law requiring that the real estate owned by a life insurance company shall be carried in the assets of the company at an amount not exceeding the capitalized value of the average net income from the property over a series of years, calculated at a specified rate of interest.

(*b*) Sketch the financial conditions which led to the adoption of the amortization method of valuing the bonds owned by life insurance companies and state what consideration in amortizing securities you would give to (1) an option of redemption; (2) an option of conversion; (3) income bonds; (4) bonds in default.

12. Discuss the advantages and disadvantages of each of the following methods of dealing with under-average lives:

(1) Rating-up applicants by adding a certain number of years to real age;

(2) Issuing policies at regular rates but making a decreasing deduction from the sum insured;

(3) Issuing policies at regular rates but placing them in special dividend classes as regards mortality;

(4) Issuing policies at rates based upon a percentage addition to the rates of mortality shown in a standard table.

13. Discuss the treatment you would accord to applicants for insurance belonging to the following types as regards special hazard:

(1) Hazardous occupation;
(2) Residence in unhealthy climate;
(3) Family history of tuberculosis;
(4) 30% over average weight for age.

14. (*a*) A policy is issued on the joint lives of John Doe and Richard Roe and assigned to John Doe and Richard Roe, Incorporated. The corporation becomes bankrupt before the policy has a cash value. John Doe wishes to change the policy to one on his life alone. What papers should the company require to be completed?

(*b*) What points must be taken into consideration in order to preserve the equities of the insured and the company, and how should the agent's commission account be adjusted, in making policy-changes of the following types?

(1) From a lower to a higher-premium plan for the same amount;
(2) From a higher to a lower-premium plan for the same amount;
(3) From a higher to a lower-premium plan for an increased amount where the annual premium remains the same.

15. A widow aged 50 is entitled for life to the income from her husband's estate comprising $10,000 (par and market value) in 4% railroad bonds and $20,000 in real estate mortgages at 5%. Her daughter, aged 25, is entitled to a maintenance allowance of $500 per annum as a first charge upon the estate and in addition is absolutely entitled to the reversion upon the death of the widow. It is agreed by both parties to make an immediate division of the estate. How should it be apportioned?

16. (*a*) State the considerations that enter into the question of restoring to full force a policy now being carried under extended term insurance.

(*b*) An insured writes to the company that he has lost his policy, and, further, that his age was understated in his application. The beneficiary under the policy is his wife and no

assignment is on record. Draft a form for execution in connection with the issuing of a new policy at correct age which will fully protect the company.

(*c*) Assuming that you were retained to consult with a legislative committee upon a proposal to require by statute that the cash surrender value of every life insurance policy shall be at all times equal to the reserve thereon, outline the considerations which you would bring to the attention of the committee.

PART II.

1. (*a*) It has been decided to calculate the annual dividends of a company as nearly as possible in accordance with the following formula:

$${}_nB_x=(i'-i)({}_{n-1}V_x+P_x)+s'(P'_x-P_x)+(q_{x+n-1}-q'_{x+n-1})(1-{}_nV_x)$$

Interpret the formula and state how you would in practice determine the factors i', s' and q'.

(b) Compare the advantages and disadvantages of the foregoing formula with those of the following:

$${}_nB_x=(i'-i)\,{}_nMV_x+r'(P'_x-P_x)$$

2. (*a*) Distinguish between the Sprague, Lidstone, and Browne modifications of the contribution method of apportioning surplus.

(*b*) Outline the process of analyzing the accounts of a life insurance company which has deferred dividend policies on its books, for the purpose of determining the factors needed for the calculation of the asset-shares of policies in this class.

3. (*a*) Explain how you would assess the following classes of expenses for the purpose of surplus distribution:

(1) Expenses of investigating and adjusting death claims;
(2) Agency supervision and traveling expenses of supervisors;
(3) Home office salaries;
(4) Rent;
(5) Advertising;
(6) The Federal Corporation Tax (in the case of a United States Company).

(*b*) In apportioning surplus, should the mortality experience of the entire company be used, or should separate account be taken of the experience in different sections of the business? Cite authorities in support of your opinion.

(*c*) How would you answer the above question as relating to the particular case of an assessment association reorganized on a legal reserve basis but retaining a portion of the assessment business with unchanged contracts?

4. (*a*) Discuss the extent to which tables giving the following ratios calculated from the annual statements of a large number of life insurance companies for a series of years may be assumed to afford valid and useful information:

(1) Actual to expected mortality;

(2) Net interest earned to mean gross assets;

(3) Net interest earned to mean liabilities requiring interest.

Assuming that it is decided to supplement the above data with a table of expense ratios, suggest the most suitable basis.

(*b*) A small company with a limit of $10,000 on a single life wishes to enter into a contract for the reinsurance of all risks in excess of that amount. Suggest the various considerations which should enter into the negotiation of a reinsurance treaty suitable to the case.

5 (*a*) Draft a form of receipt for a sum of money received from an applicant for insurance to be applied on the first premium in case the policy is issued.

(*b*) On December 27, 1912, the application of John Doe for $5,000 of insurance at an annual premium of $170.25, accompanied by John Doe's check for $170.25 in conditional payment of the first premium, is received at a branch office and the money deposited in bank subject to draft by the home office. Notice of this transaction is received at the home office on December 30. On January 8, 1913, the application is rejected and on January 11, on presentation of his interim receipt at the branch office, the applicant receives the company's check for $170.25. Indicate the necessary entries in the home office books and the effect of the transaction on the company's statement as of December 31.

6. (*a*) Summarize the advantages and disadvantages of farm loans as investments for life insurance companies.

(*b*) Enumerate the points to be covered in a form of application for a farm loan.

(*c*) Describe the methods of operation of the Credit Foncier of France and of the Landschaften of Germany.

7. (*a*) Define the following terms as ordinarily used in American railroad reports:

(1) Gross earnings;
(2) Operating expenses;
(3) Net earnings;
(4) Total net income;
(5) Fixed charges.

What may be learned as to the desirability of the securities of a railroad from the ratio of the total fixed charges to the total net income?

(*b*) State the main points to be considered in judging, as investments for life insurance companies, public service corporation bonds of the following classes:

(1) Electric light and power;
(2) Gas;
(3) Steamship;
(4) Street railway;
(5) Telephone;
(6) Water.

8. The trial balance of the life department of a stock insurance company, drawn up on a revenue basis as of December 31, 1912, is as follows:

ACCOUNTS	DEBITS	CREDITS
1. Death claims	$1,487,500	
2. Matured endowments	230,000	
3. Surrender values	590,000	
4. Dividends to policy-holders	865,000	
5. Dividends to shareholders	35,000	
6. Commissions and expenses of management	849,500	

7. Premiums		$5,012,000
8. Interest and rents		1,468,000
9. Real estate	650,000	
10. Mortgage loans	11,500,000	
11. Policy loans	3,500,000	
12. Bonds	13,000,000	
13. Cash	835,500	
14. Outstanding premiums	375,000	
15. Accrued interest	450,000	
16. Shareholders' capital		500,000
17. Policy reserve (Dec. 31, *1911*)		24,500,000
18. Surplus (Dec. 31, *1911*)		1,720,000
19. Outstanding claims		62,500
20. Dividend funds		950,000
21. Unearned interest		90,000
22. Outstanding expenses		65,000
	$34,367,500	$34,367,500

At the end of the year the policy reserve is $26,700,000.

(*a*) Show the journal entries necessary to close the books.

(*b*) Draw off in the general form prescribed by the (British) Assurance Companies Act, 1909, but without conversion to English currency: (I) a revenue account; (II) a profit and loss account; (III) a balance sheet.

*9-A. A large public service corporation desires to adopt a pension and retirement plan for its employees which will provide the following benefits:

(*a*) An allowance for temporary or permanent disability;

(*b*) An allowance to dependent relatives at death;

(*c*) An old-age pension.

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*Candidates who prior to the year 1910 passed Section A of the Fellowship Examination will omit Question 9-A and will take Question 9-B.

Candidates who passed Part I of the Fellowship Examination in 1910, 1911 or 1912, or who have not yet passed in Part I will omit Question 9-B and will take Question 9-A.

The remaining questions are for all candidates.

The cost of the plan is to be wholly borne by the corporation. What benefits would you recommend should be granted under each of the heads mentioned to best promote the efficiency and ultimate welfare of the employees? How would you proceed to estimate the annual cost of such a plan and what provisions would you suggest should be made with regard to a periodical valuation?

9-B. (*a*) Discuss the relative advantages to both policyholder and company of the following methods of providing for automatic non-forfeiture in the event of default in payment of premiums:

(1) Paid-up life or endowment insurance;

(2) Extended term insurance for the original amount;

(3) Premium loan.

(*b*) What rate of interest on automatic premium loans would you recommend should be stipulated in the policy? Give the reasons for your answer.

10. (*a*) Define what is meant by the "rate of exchange" between two countries and indicate the principal factors by which it is determined.

(*b*) Of what value is a knowledge of the movements of rates of domestic and foreign exchange in judging the supply of money immediately available for investment in securities?

11. (*a*) The statement of the New York Clearing House banks and trust companies for a certain week was as follows:

	AMOUNT	CHANGE FOR WEEK
Loans	$1,912,213,000	+ $14,618,000
Deposits	1,760,136,000	+ 21,548,000
Circulation	46,340,000	+ 45,000
Specie	329,464,000	+ 7,908,000
Legal tenders	81,877,000	− 961,000
Cash reserve required	395,578,400	+ 4,740,000

Interpret the statement, explain the meaning of the various items and indicate the practical value of the publication of such statements.

(*b*) When the "call rate" for money is greater than the "time rate", what does this indicate as to whether interest

rates in the immediate future will probably strengthen or weaken?

12. (*a*) Summarize those provisions of The Insurance Act, 1910, of the Dominion of Canada which relate to the valuation of the policies and the investment of the funds of life insurance companies and compare them with the statutory requirements in force prior to the passing of the Act.

(*b*) The following data are assumed to represent the facts for the year 1912 regarding a life insurance corporation admitted to transact business in the State of New York:

New business written	$175,000,000
Gross renewal premiums	65,000,000
Loading	21% of the gross
Total expenses	11,000,000
First-year expenses specified in Section 97 of the New York Insurance Law	3,500,000
Investment (including real estate) expenses	1,000,000

The company has been writing a constantly increasing business for more than three years. Calculate the maximum new business which may lawfully be written by the company during 1913 under Section 96 of the New York Insurance Law.

13. (*a*) Discuss the actuarial features of the so-called "Mobile Law" which has been enacted in several states providing, among other matters, for the valuation of certificates issued by fraternal societies.

(*b*) Describe the position taken by the courts in at least two important cases regarding the right of a fraternal society to increase its rates of assessment.

14. (*a*) Describe the existing statutes in either New York State or Canada providing for standard policy provisions.

(*b*) Compare the advantages and disadvantages of a "standard provision" law with a "standard policy" law.

15. (*a*) Describe the benefits ordinarily provided in the personal accident and health policies issued by casualty companies in the United States and Canada.

(*b*) How are the premiums and reserves on such policies computed?

16. (*a*) State what statistics are available as a basis for computing sickness insurance premiums and show how you would compute premiums for a sickness benefit of 1 per week being given a table of adjusted rates of sickness.

(*b*) What fundamental data and formulae would you use in calculating rates for waiver of premium in event of total and permanent disability occurring before age 60.

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TRANSACTIONS

OCTOBER 23D AND 24TH, 1913.

THE INCOME TAX LAW OF THE UNITED STATES, AS IT AFFECTS LIFE INSURANCE COMPANIES.

BY

E. E. RHODES.

The Income Tax Law enacted during the present special session of Congress comprises section II of an act "To reduce tariff duties and to provide revenue for the Government, and for other purposes," approved October 3, 1913. There are fifteen paragraphs, or subsections, bearing in successive order the first fifteen letters of the alphabet. Paragraphs *A–F*, inclusive, deal with the tax imposed upon individuals. There is, however, a provision in paragraph *B*, which, in my judgment, affects the tax imposed upon corporations as well as individuals, and which will be referred to hereafter.

We shall first consider the manner in which the amount of tax payable by life insurance companies organized in the United States is to be determined. The main provisions therefor are found in paragraph *G*.

As set forth in subdivision (*a*) of paragraph *G*, "the normal tax (one per cent.) hereinbefore imposed upon individuals likewise shall be levied, assessed, and paid annually upon the entire net income arising or accruing from all sources during the preceding calendar year to every . . . insurance company." Subdivision (*b*) provides that "such net income shall be ascertained by deducting from the gross amount of the income of such . . . insurance company, received within the year from all sources, (first) all the

ordinary and necessary expenses paid within the year in the maintenance and operation of its business and properties, including rentals or other payments required to be made as a condition to the continued use or possession of property; (second) all losses actually sustained within the year and not compensated by insurance or otherwise, including a reasonable allowance for depreciation by use, wear and tear of property, if any; . . . and in case of insurance companies the net addition, if any, required by law to be made within the year to reserve funds and the sums other than dividends paid within the year on policy and annuity contracts; Provided, that . . . life insurance companies shall not include as income in any year such portion of any actual premium received from any individual policyholder as shall have been paid back or credited to such individual policyholder, or treated as an abatement of premium of such individual policyholder, within such year; (third) the amount of interest accrued and paid within the year on its indebtedness . . . ; (fourth) all sums paid by it within the year for taxes imposed under the authority of the United States or of any State or Territory thereof, or imposed by the Government of any foreign country."

Paragraph *B* provides, "That in computing net income under this section there shall be excluded the interest upon the obligations of a State or any political subdivision thereof, and upon the obligations of the United States or its possessions." While paragraph *B* generally relates to the net income of taxable persons, the language of this provision seems broad enough to cover the net income of corporations also, and this appears to have been the intention of Congress. The income tax provisions of the present act are comprised in one section, which is subdivided into subsections or paragraphs. If Congress had not intended that the rule quoted above should apply to corporations as well as persons it would have used the word "paragraph" or "sub-section" instead of the word "section," as it consistently did elsewhere when it intended to limit specific provisions to the net income of either individuals or corporations. It will be borne in mind that there is a very important distinction between the present act and the act of August 5, 1909. The latter act was not a tax upon incomes but an excise tax, the income being the measure of the tax. The present act does not impose a tax "with respect to the carrying on or doing business," but does impose a tax upon net incomes, and the reasons

which led to the exclusion, from the net income of persons, of interest upon the obligations of the United States or of a state or political subdivision thereof, apply with the same force to the net income of corporations.

In preparing the return of net income it will be found desirable, in so far as possible, to follow the return required by the insurance department of the state in which a company is located, eliminating those items which do not represent actual income or outgo and which are no essential part of the official return, but which are called for by insurance departments to enable them to be informed of the various circumstances of the life insurance business. Using the form of return prepared by the National Convention of Insurance Commissioners for the year ending December 31, 1912, the gross income will embrace the following items: first year's premiums on original policies, not including surrender values applied to pay first year's premiums; first year's premiums on annuities; consideration for supplementary contracts involving life contingencies; renewal premiums on insurances (contract premiums less amounts allowed in abatement thereof); renewal premiums on annuities; interest, dividends and rents, less "interest upon the obligations of a State or any political subdivision thereof, and upon the obligations of the United States or its possessions"; profit on sale or maturity of ledger assets; increase by adjustment in book value of ledger assets; income from other sources. The item of "dividends paid to policyholders in cash" appearing in the disbursements would include certain dividends (such as those on deferred dividend policies or the final dividend on a limited premium annual dividend policy in its premium-paying form) constituting a portion of the actual premiums received and paid back or credited to individual policyholders, and the amount thereof should be deducted from the gross income ascertained as above. The same would be true of dividends on premium-paying policies left with the company to accumulate at interest.

The "ordinary and necessary expenses" paid within the year will include: expense of investigation and settlement of policy claims; commissions to agents; commuted renewal commissions; compensation of managers and agents not paid by commission; agency supervision and traveling expenses; branch office expenses; medical examiners' fees and inspection of risks; salaries; rent; advertising, printing and stationery; postage, telegraph, telephone and express;

exchange; legal expense, not including that incurred in investigation and settlement of policy claims; furniture, fixtures and safes; repairs and expenses, other than taxes, on real estate; miscellaneous expenses.

The "losses actually sustained within the year, including a reasonable allowance for depreciation," will include: agents' balances charged off; loss on sale or maturity of ledger assets; decrease by adjustment in book value of ledger assets.

The "net addition required by law to be made within the year to reserve funds" will be the difference between the reserve on original contracts and on supplementary contracts, whether or not involving life contingencies, at the end of the year for which the return is made, and the corresponding reserve at the end of the previous year.

The "sums other than dividends paid within the year on policy and annuity contracts" will comprise: death claims and matured endowments, decreased by the "consideration for supplementary contracts not involving life contingencies" and increased by the amount paid for claims on such supplementary contracts; annuities involving life contingencies; premium notes and liens voided by lapse; surrender values paid in cash.

The "interest accrued and paid within the year on indebtedness" will consist, in addition to interest upon other indebtedness, if any, of interest at the agreed rate upon the proceeds of matured policies retained by a company for the benefit of the insured or beneficiary in accordance with the provisions of the policy contracts. It would not include any part of the payments under instalment contracts, deductible under the first item in the preceding paragraph.

The "sums paid within the year for taxes" will include all licenses and fees imposed by law, as well as all taxes of whatsoever character.

Under the act of August 5, 1909, it was held in the case of Mutual Benefit Life Ins. Co. *vs.* Herold, Collector (198 Fed. 199, and 201 Fed. 918), that the return should be made on a "cash" and not a "revenue" basis; that the reserves for supplementary policy contracts are a part of the company's reserve fund required by law; and that expenditures for replacing furniture, etc., should be considered as an expense. It was also held that the so-called dividends allowed in abatement of premiums by life insurance companies

doing business on the mutual plan are not "income" "received," but this question is not involved in the Income Tax Law, in which it is expressly provided that such dividends are not to be included in income.

The writer would suggest that there be annexed to the returns a schedule showing in detail the several items entering into the gross income, and the deductions therefrom. Such a schedule will not only enable the authorities to readily verify the returns, but will avoid any charge of intentional concealment or evasion.

Subdivision (*c*) of paragraph *G* provides that the return shall also show "the total amount of its paid-up capital stock outstanding, or if no capital stock, its capital employed in business, at the close of the year"; also "the total amount of its bonded and other indebtedness at the close of the year." A life insurance company has no bonded indebtedness, but it has other indebtedness, unless the word "other" is to be construed as meaning indebtedness of the same general character as bonded indebtedness, which seems unlikely. If it be held to mean all indebtedness, the total liabilities may be returned; or the item may be limited to such liabilities as are presently payable, namely, matured policy claims, commissions to agents due or accrued, salaries, rents, office expenses, bills and accounts due or accrued, medical examiners' fees and legal fees due or accrued. The question is unimportant except as it affects the amount claimed as a deduction on account of interest paid on indebtedness. The amount of such deduction must not exceed the interest on one-half of the sum of the interest bearing indebtedness, and paid-up capital stock outstanding at the close of the year, or if no capital stock, the interest on an amount of its indebtedness not exceeding the amount of capital employed in the business at the close of the year. In a mutual company, presumably, the "capital" employed in the business is at least the undivided funds.

The additional tax levied upon individuals for incomes in excess of twenty thousand dollars is not imposed upon corporations, and the exemptions and deductions which apply to individuals do not apply to corporations, except as above.

For the year ending December 31, 1913, the tax is to be ascertained by taking five-sixths of the entire net income for said calendar year.

Paragraph *S* of section IV of the act provides that the corporation tax imposed under section 38 of the act of August 5, 1909,

shall be levied for the period from January 1 to February 28, 1913, and that said tax shall be computed upon one-sixth of the entire net income for said year ascertained in accordance with the provisions of subsection *G* of section II of the present act. The act of August 5, 1909, provided for the deduction of five thousand dollars from the entire net income. Otherwise the taxable net income of insurance companies would be ascertained under the new act in the same way as under the former act, with the exception that under the new act the interest upon the obligations of a state or any political subdivision thereof, and upon the obligations of the United States or its possessions, is not included in the income, if my construction of the law is correct, and certain dividend credits, paid in cash, referred to above, are to be deducted from the gross income. Applying the provisions of the present act to the taxable income for the period from January 1 to February 28, 1913, as provided for in paragraph *S* of section IV, the companies lose the benefit of a pro-rata deduction in the amount of five thousand dollars, but in addition to the dividend credits mentioned, receive, in my opinion, the benefit of a pro-rata deduction of the interest upon obligations of the United States or of a state or political subdivision thereof, the income from which was taxable under the act of August 5, 1909.

It will not be necessary to make more than one return of net income for the year ending December 31, 1913.

Under subsection (*c*) of paragraph *G*, an insurance company may designate the last day of any month in the year as the day of the closing of its fiscal year, and be entitled to have the tax payable by it computed upon the basis of the net income for the year ending on the day so designated.

Subsection (*c*) of paragraph *G* provides that the return of net income shall be filed with the Collector of Internal Revenue for the District in which the company has its principal place of business, on the first day of March each year, or within sixty days after the close of its fiscal year. All assessments are to be made and the companies notified of the amount for which they are liable on or before the first day of June, or within sixty days after the return is made, and the assessment is to be paid on or before the thirtieth day of June, or, if the fiscal year be other than the calendar year, within one hundred and twenty days after the date upon which it is required to file its return.

The net income of an insurance company organized outside of the United States but doing business therein is based upon business transacted within the United States.

Fraternal beneficiary societies, orders, or associations operating under the lodge system or for the exclusive benefit of the members of a fraternity itself operating under the lodge system, and providing for the payment of life, sick, accident, and other benefits to the members of such societies, orders, or associations, and dependents of such members, are not subject to an income tax.

It is not deemed necessary to refer to the penalties for failure to make returns or for false or fraudulent returns.

II.

The act contains provisions for the collection of income taxes of individuals at the source of income, which will now be noted in so far as they relate to payments of life insurance companies. These provisions should be clearly understood, as, under paragraph *E*, insurance companies are made liable for such tax on all funds in their hands from which the tax should be deducted and withheld. These provisions do not apply to the tax imposed upon corporations (paragraph *E*).

Paragraph *B* provides "that the proceeds of life insurance policies paid upon the death of the person insured or payments made by or credited to the insured, on life insurance, endowment, or annuity contracts, upon the return thereof to the insured at the maturity of the term mentioned in the contract, or upon surrender of contract, shall not be included as income."

Paragraph *D* provides that "all . . . insurance companies, except as hereinafter provided, in whatever capacity acting, having the control, receipt, disposal, or payment of fixed or determinable annual or periodical gains, profits, and income of another person subject to tax, shall in behalf of such person deduct and withhold from the payment an amount equivalent to the normal income tax (one per cent.) upon the same and make and render a return, as aforesaid, but separate and distinct, of the portion of the income of each person from which the normal tax has been thus withheld, and containing also the name and address of such person . . . ; Provided, That the provision requiring the normal tax of individuals to be withheld at the source of the income shall not be construed to require any of such tax to be withheld prior to the first day of

November, nineteen hundred and thirteen: Provided further, That in either case above mentioned no return of income not exceeding $3,000 shall be required."

Paragraph *E* provides that "All . . . insurance companies . . . having the control, receipt, custody, disposal, or payment of interest, rent, salaries, wages, premiums, annuities, compensation, remuneration, emoluments, or other fixed or determinable annual gains, profits, and income of another person, exceeding $3,000 for any taxable year, other than dividends on capital stock, or from the net earnings of corporations and joint stock companies or associations subject to like tax, who are required to make and render a return in behalf of another, as provided herein, to the collector of his, her, or its district, are hereby authorized and required to deduct and withhold from such annual gains, profits, and income such sum as will be sufficient to pay the normal tax imposed thereon by this section, and shall pay to the officer of the United States Government authorized to receive the same; and they are each hereby made personally liable for such tax. In all cases where the income tax of a person is withheld and deducted and paid or to be paid at the source, as aforesaid, such person shall not receive the benefit of the deduction and exemption allowed in paragraph *C* of this section except by an application for refund of the tax unless he shall, not less than thirty days prior to the day on which the return of his income is due, file with the person who is required to withhold and pay tax for him, a signed notice in writing claiming the benefit of such exemption and thereupon no tax shall be withheld upon the amount of such exemption: . . . nor shall any person under the foregoing conditions be allowed the benefit of any deduction provided for in subsection *B* of this section unless he shall, not less than thirty days prior to the day on which the return of his income is due, either file with the person who is required to withhold and pay tax for him a true and correct return of his annual gains, profits, and income from all other sources, and also the deductions asked for, and the showing thus made shall then become a part of the return to be made in his behalf by the person required to withhold and pay the tax, or likewise make application for deductions to the collector of the district in which return is made or to be made for him:"

The word "annuities" in paragraph *E*, quoted above, does not appear to apply to regular annuity contracts of life insurance com-

panies, payments under which are not to be included as income, as specifically provided in paragraph *B*.

Paragraph *E* further provides "That the amount of the normal tax hereinbefore imposed shall be deducted and withheld from fixed and determinable annual gains, profits, and income derived from interest upon bonds and mortgages, or deeds of trust or other similar obligations of corporations, joint stock companies or associations, and insurance companies, whether payable annually or at shorter or longer periods, although such interest does not amount to $3,000, subject to the provisions of this section requiring the tax to be withheld at the source and deducted from annual income and paid to the Government."

With regard to payments falling due under policy contracts, it is clear that the proceeds of life and endowment policies, whether paid in one sum or in instalments, and payments falling due under annuity contracts, and surrender values, are not taxable under the proviso in paragraph *B* quoted above. Dividends paid upon deferred dividend policies at the completion of the dividend period are also not taxable inasmuch as they are "payments . . . credited to the insured on life insurance, endowment, or annuity contracts, upon the return thereof to the insured at the maturity of the term mentioned in the contract." It seems clear that the phrase "term mentioned in the contract" relates to the contract provisions regarding dividends, and I am of the opinion that under this proviso the annual dividends upon paid-up policies are not taxable, especially where the policy contract contains a provision for the payment of such dividends. Inasmuch, however, as such paid-up dividends are paid out of the net earnings of insurance companies subject to the income tax imposed thereon, they would appear, as set forth elsewhere in paragraph *B*, as a deduction in a personal return. In any event, no tax thereon need be withheld by the companies, as dividends paid out of net earnings are expressly excepted from the operation of paragraph *E*.

Subject to the provision in the act which will be referred to hereafter, the normal tax must be deducted from all salaries or other compensation paid in excess of $3,000 per annum. No difficulties present themselves in the case of salaries, but embarrassing questions are likely to arise in connection with other forms of compensation. For example, general agents who are paid by commissions and who employ soliciting agents, who in turn are compen-

sated on a commission basis by the general agents, receive from their companies comparatively large sums of money, the larger part of which, however, although paid to the general agents, really belongs to their subordinate agents. If the companies retain the normal tax on the full amount payable to general agents, they will have in their hands a very much larger sum than will be due the Government, and the retention thereof may embarrass those general agents whose resources are limited. General agents will not be liable for a tax on that part of their commissions paid them by the companies which in turn is paid by them to soliciting agents, as such commissions are deductible as a part of the "necessary expenses actually paid in carrying on any business," in conformity with paragraph *B*.

I am inclined to think that the companies will be sufficiently within the law, if, instead of deducting the normal tax upon the gross sums paid to general agents from time to time during the year, they wait until the general agent files with them his return of net income, when the net amount due the government is ascertainable, or until application for exemption is made to the collector for the district in which return is to be made for him. If this plan is followed it would be well if the contracts between the company and the general agent provided that if the company did not have in hand sufficient funds of the general agent to pay the tax, the company might use so much of the present value of the general agent's equity in future renewal commissions as might be required for that purpose.

The proviso in paragraph *E*, quoted above, regarding "fixed and determinable annual gains, profits, and income derived from interest upon bonds and mortgages, or deeds of trust or other similar obligations of corporations, joint stock companies or associations, and insurance companies" requires some comment. The language of the bill as introduced and as it passed the House was "income derived from interest upon bonds, mortgages, or other indebtedness." In the Senate this language was modified to read "interest upon bonds and mortgages, or deeds of trust, or other obligations." In conference a further amendment was made, the word "similar" being inserted between the words "other" and "obligations," thus further restricting the meaning. Bonds and mortgages or deeds of trust are obligations secured by collateral. The word "obligations" as here used has a different meaning from the word "indebtedness." It may properly be construed to refer only to such indebtedness as is

evidenced by the particular kinds of obligations mentioned, and those of a similar kind, viz.: secured by a mortgage or pledge of property as collateral security. Life insurance companies generally have no such obligations, and the provision, therefore, does not affect them.

Paragraph *D* provides that a personal return must be made "to the collector of internal revenue for the district in which such person resides or has his principal place of business," and that "any person for whom return has been made and the tax paid, or to be paid, as aforesaid, shall not be required to make a return unless such person has other net income, but only one deduction of $3,000 shall be made in the case of any such person." It would appear that if the income of a person exceeds $3,000, and the tax on some part, but not the whole, thereof is withheld at the source, as provided in paragraph *E,* two returns must be filed, one as above provided (paragraph *D*) ; and (preliminary to obtaining the benefit of any deductions under paragraph *B*) the other with the party withholding the tax, or with the collector for the district in which such withholding party resides (paragraph *E*). If, however, a signed notice in writing is filed within the required time with the person required to withhold the tax, claiming the benefit of the exemption in paragraph *C,* no tax need be withheld upon the amount of such exemption, and presumably, if the income is less than the amount exempted in paragraph *C,* no return of income need be filed by the taxable person. The normal tax on the income in excess of such exemption, however, would have to be withheld, and two returns filed by the taxable person, unless the return, at source, covers his entire income.

It will be noted that the law does not specifically state that the tax need *not* be withheld, when the deductions allowed under paragraph *B* are claimed, as is the case when the exemption allowed in paragraph *C* is claimed.

Some time will doubtless elapse before the Government will be in a position to make authoritative rulings upon the questions which will arise in connection with the law. The opinions expressed above represent only the personal views of the writer.

Charles Gill; the First Actuary in America.

(Second Paper.)

BY

EMORY McCLINTOCK.

Some further facts concerning Gill's life in England have come to light recently which I owe to the care of my friend, Mr. S. A. Joffe, the author of several papers on subjects connected more or less with the work of Mr. Gill.

After the May meeting, I wrote Mr. Joffe for copies of Gill's manuscripts from the office of the Mutual Life, as well as for anything else about Gill which might be in his power to give me. He at once replied that he had noticed Gill's name in some numbers of the *Ladies' Diary,* as having solved problems in the years 1823–30; namely, in 1823–24 dated from "Ganton, near Scarbro"; in 1826–27 dated "Hessle, near Hull," and in 1829–30 dated from "Seamer, near Scarboro." Afterwards, when I had the pleasure of a visit from him during his vacation, I remarked that the application indexes of the Mutual Life might profitably be consulted for further clues about Gill, as we had the names of his children. On his return, he sent me copies of two applications of Eliza Anne Gill from which the dates of birth were found or inferred as follows: Mrs. Eliza E. Gill, 1805; Eliza Anne Gill, October 3, 1828, and Wm. C. Gill, 1842. It appears that the youngest daughter, Mary Gill, died very soon after her father of smallpox, and that the other children, first named, may possibly be still living in advanced life.

These few facts, with some others, as data, and some slight help from the imagination will be found to give a greatly extended idea of his early life, although I am not much in the habit of drawing on my imagination in the endeavor to fill out the gaps in my data. Seamer and Ganton may be taken first. They are townships on the high road south from Scarborough, one on each side of Staxton, Gill's birthplace. Seamer is about four miles southwest of Scarborough, and Staxton is two miles farther south. The road not being very straight, Ganton is about three miles further from Scarborough, westerly from Staxton. Gill was only seventeen when

he taught school first at Ganton, and this was his start in life as a mathematician.

GANTON.

To get one's name in the *Ladies' Diary* for 1823, one had to buy or borrow a copy of the *Diary* for 1822, this publication being issued only once a year, presumably on January 1. As Gill was born on or after October 20, 1805, he must have been a teacher at Ganton before October 20, 1822, when he was seventeen years of age. The Staxton school must have been good for little, or even good for nothing. It was probably a part of the religious outfit of the parish of Willerby, Staxton being two or three miles from the church, and the school may have been located nearer the church. There was, at that time, no regular system for teaching the children of any parish, the school being usually one of the benevolent duties of the parish priest, or minister, as he was usually called at that pre-Puseyite period. There was also a Wesleyan chapel at Staxton, and Gill might have been brought up, as a child, either in church or chapel, but the presumption is that he was a churchman. He is represented by Dr. Muhlenberg to have been, throughout his life, deeply religious.

The finding of employment, for the first time, was a thing of difficulty to this boy of sixteen. He had just returned from his dangerous trip to the West Indies, and may be considered a real hero in being able to assume the command of the vessel at a time when the officers had all died from pestilence. (It will be recollected that when he was thirteen, he took a place as an able seaman, which may be interpreted as running away from home: he was said to have learned what he could from the teacher at Staxton at ten years of age.) It was a critical time in his life. Turned loose on the world at that age, with such surroundings, the "making of a spoon" might readily enough have turned out to be the "spoiling of a horn." No word is heard in any quarter about his father, nor of his mother. His silence about his parents is remarkable.

How did he come to be employed in the parish school of Ganton, near by, at the age of sixteen? Character he certainly had, and hardships could only have served to bring out the stuff that was in him. What character had he? Two points, each great in itself. One was a self-reliance that was his Yorkshire birthright (I who write this know what that means, for I spent three years and more in that same county of York and know what the sturdy spirit of self-

dependence is, which is born in every Yorkshireman); the other was his extreme ambition. These traits, with the great foundation of religion, were the chief characteristics of Charles Gill. But character alone was not enough. Who recommended him? Certainly his appointment could not have come from any powerful influence exercised by his old teacher at Staxton, who might have given him a good word, but little more.

Of course, he went about among his friends and neighbors, and may have spent some time in doing so. The nearest man who could have helped him was however the teacher of the school at Seamer, which was a parish on the same road about two miles from Staxton northwardly. I cannot find out his name, so will write of him as X. The school-master, X, was a favorite of the lord of the manor of Seamer, a Mr. W. J. Denison, possibly a relative of his. I have a topographical dictionary issued in 1831, giving an article about every place in England. The article on Seamer states a remarkable thing about Mr. Denison, considering the year of which it is written, 1814. Mr. Denison laid out a large amount of money in that year in building schools for boys and girls and endowing them "liberally"; under this endowment the head-master appears to have been provided with a good place for life, since one item is said to have been a house for his use.

When Gill was sixteen, after the striking evidence of his youthful energy which was given when he assumed command of the apparently derelict vessel, Mr. X must have heard of the lad as the hero of the neighborhood, because Staxton was only two miles away, the nearest village. Mr. X was the great man of the vicinity as a schoolmaster, and may be supposed to have been himself better educated than any other teacher for miles around. When Gill was on his tour of the neighborhood looking for a place, Mr. X, if he did not know him already, was the first man whom he could approach, and we may be sure Gill had not a particle of bashfulness in his composition. I do not draw too far upon my imagination when I say that Mr. X was the very man to become his helper in this time of his real need. I assume that his help was forthcoming, and that Gill's getting the place at Ganton was due to his influence.

The *Ladies' Diary* for 1822 must have been beyond his means as a subscriber. There was probably no subscriber to it, except Mr. X, nearer than Scarborough, if, indeed, any more were to be found there. This was then a rather costly part of a teacher's outfit, and

if he subscribed in 1823, the price must have been a heavy tax on Gill's meagre salary. Before the end of 1823, Gill sent in answers to ten of the fifteen problems for that year, and he did perhaps better than ten, if we count quality as well as quantity. We are able to do so, thanks to the copy of the problems themselves sent me by Mr. Joffe for the purpose of comparing their value. They are inserted verbatim in an appendix. (See Appendix *A*.)

Of the fifteen questions given in 1822 for 1823 there were thirteen which may be considered geometrical, and two analytical. The two were Nos. 14 and 15, neither of which was answered by Gill. On the other hand, Gill's answer to the thirteenth question was not only given in full, but is singled out by the editor as positively the best. Considering that Gill entered the lists at the age of eighteen, though his age does not appear to have been stated, his answers to the questions contained in the *Diary* of 1823, printed in the *Diary* of 1824, must have made a sensation in the office. The *Diary* of 1825 shows his answers to the questions in that of 1824. Questions No. 13 and No. 14 remained unanswered by him, as did the last two of the year before, No. 13 being astronomical in its character, while No. 14 is equally or more difficult. Again his replies to two questions are printed in full, as before, with additions which gave the editor pleasure to insert. To No. 4, in the *Diary* for 1824, his answer was thought worthy of being stated as a new problem. In answering No. 10 he was in distinguished company, the same answer being given by Professor Thomson of the College of Belfast, the father of Lord Kelvin (*clarum ac venerabile nomen*).*

The *Ladies' Diary* for 1823 records that Professor James Thomson of the College of Belfast, and author of a treatise on arithmetic, proposed question No. 14, while that for 1825 says that Professor Thomson (obviously the same) and Mr. Charles Gill have answered No. 10 differently from a youth of fourteen, signing himself "W. S. B. Woolhouse." The identity of this precocious mathematician, burdened with those initials, cannot be mistaken by any reader of the *Journal of the Institute of Actuaries* of the past generation. I, for one, recognize the name as that of one who was to become famous as a Senior Wrangler at Cambridge and later as a prominent actuary usually called upon for any work of the Insti-

* I met Lord Kelvin at the Second International Congress of Actuaries, in which he was actively interested. He was presiding officer at the great dinner, and I had the honor of sitting next to him.

tute clearly requiring mathematical genius. The mention of Gill's answer, as being identical with that of Professor Thomson, was equal to giving him distinct rank as a first-class mathematician, and should have at once won him a higher post as a teacher.

Hessle.

The *Diary* shows him next teaching at Hessle, an important suburb of Hull, which was an older place than Hull, and was once so jealous of its rapidly growing neighbor as to become the subject of a lawsuit. Hull was named Kingston—or King's town—being given special privileges by its founder, King Edward I, and Hessle and one or two parishes near by were accused of a conspiracy against the drinking water of their upstart rival. In 1831 Hull had a population of 30,000 while Hessle had 1,021, the places being five miles apart. Hessle had a Wesleyan chapel as well as a church, and also had a national school besides the usual parish school. In which of them Gill taught cannot be told, though the national school was probably his choice, as the authorities of a public school would be more likely to employ a stranger, as he was, aided by a commendatory letter.

Gill may have left Ganton in 1825 or 1826, as none of the problems of the *Diary* of 1825 were answered by him. In 1825 he must have been too poor to pay the cost of subscription. I assume that he was too poor previously, because his first place was obviously the hardest. Mr. X must have given the use of his copies for two years, unless Gill had found some mathematical friend at Scarborough to lend them to him. Dr. Muhlenberg especially says, in speaking of Gill's poverty during his early years as a teacher, that he forebears "to relate what a long course of self-denial and relinquishment of the dearest enjoyments of life (for a number of years, indeed) he subjected himself to, in order to wipe off some pecuniary obligations, occasioned by the difficulties of his early career." Such a postponement of matrimony is not unusual, though the case must have been a peculiarly hard one. I assume that Gill made the acquaintance at Ganton of the young lady whom he married in Hessle in 1827 or 1828. Her name was Eliza E. ———, surname unknown. Hessle must have been a barely endurable place to live in, but Hull, near by, had some advantages. It was really a large city and contained, doubtless, many men of studious turn, of like views with himself. In 1822 there had been formed in Hull a

"Literary and Philosophical Society," which probably "took in" an annual copy of the *Ladies' Diary* for its reading room or library. Assuming that it did, Gill could better afford to pay the annual dues to the society than the charge, whatever it may have been, for a year's subscription to the *Diary*. In 1826 Gill answered for the *Diary* of 1827 seven of the fifteen questions, although—this being obviously the year for "Wunderkinder" as Mr. Joffe points out to me—Woolhouse, aged fifteen, and William Jacobs, aged fourteen, answered every question in 1826. Gill answered in 1828 none of the questions given out in that year, thus repeating the experience of 1825. Of his marriage in 1827 or 1828 I have thus far failed to discover the parish record, though I hope for news of it up to the last moment. I assume that every time Gill changed his school the change was for the better. Hessle had a considerably larger population than Ganton, and when he moved from Hessle to Seamer, where he doubtless occupied the comfortable house vacated by Mr. X, his best friend, he went, figuratively speaking, from a cottage to a palace.

Seamer.

Mr. X, who was installed by Mr. Denison at Seamer, when he built the schools there in 1814, had lived in the house erected for his use, until he moved to a better school in 1828. Gill was promoted to the best place he had before he left England, namely Seamer, when he was 23 years of age. The *Diary* for 1830 shows Gill to have answered in the previous year fourteen of the fifteen questions. That of 1831 shows him to have replied to thirteen out of fifteen. The years of Gill's stay at Seamer were years of prosperity and comfort. Not only was he able to free himself from debt, but he was also able to defray the cost of taking himself to New York with his family, which had been increased by the birth of a daughter on October 3, 1828. This, his first child, was named Eliza, after her mother.

For once in his lifetime everything seemed to turn out for the best. His reputation was made. The mathematicians of Great Britain, those of them, at least, who cared to enter the competition opened by the *Diary,* were represented during these ten years by several individuals who gained the highest honor, some in one year, some in another. I can say, looking over the list, that Gill's average was exceedingly high. Woolhouse and one or two others made a better average, but on the whole it may be stated with confidence

that Charles Gill was the peer of any. Books he had none, or so few as to be laughable in our eyes, and the fact that he was able to meet the requisites of the problems in the *Diary* was almost beyond credibility. I am only a cataloguer in writing this paper, I feel greatly the loss of the flowing words of some better writer than myself who would paint in proper terms the career of this brilliant young man from seventeen to twenty-five. The catalogue must suffice for the present, and the whole scene as it appears to one may be left for someone else to do justice to. We leave Gill at the height of his good fortune in England to embark on his fateful voyage with his little family to try his luck in the new world.

Saw Pits Academy.

Gill himself stated, when applying for insurance, years afterwards, to the Mutual Life, "I came to the United States January 1st, 1830." He at once sought employment. He took a place in Staten Island, which was a temporary refuge for want of a better, and exchanged it speedily for a more permanent position in Westchester County. This better place, which he found, according to Mr. Winston's statement, in a surprisingly short time—said to be ten days from his landing—was in a school, called Saw Pits Academy, which Mr. Joffe has located with great trouble in the village of Port Chester, now a part of the town of Rye. This school is mentioned in Scharf's *History of Westchester County* (Vol. 2, p. 698), which states: "This village (Port Chester), at the mouth of Byram River, was known early in the eighteenth century, and until within the last fifty years, as 'Saw Pit.' That very inelegant name had its origin in the fact that a spot on Lyon's Point, now part of the village of Port Chester, was occupied in ancient times for the building of boats. There was a landing here, known as 'Saw Pi*tt* landing,' as early as 1732; and in 1741 we hear of 'some small lots lately laid out at the Saw Pits, so called.' . . . The Saw Pit school-house stood on the west side of King Street, about forty rods from the railroad." Another school, worth mentioning in this connection, was in Rye proper and is referred to on p. 694: "There was a school-house a few years since on Regent Street, where a small office now stands (1886), not far from the corner of Purchase Avenue. Here one Evans B. Hollis taught school some fifty years ago. He was an Englishman, and is said to have been an excellent teacher." We

have seen the name of "Evans Hollis, Esq., Rye, New York," as the last of the subscribing supporters of the *Mathematical Miscellany,* started by Gill in 1836.

Gill himself was at an earlier day interested in an American publication, called the *Mathematical Diary,* which was first published in New York in 1825, and which continued until 1832. The title was one to recall the English *Ladies' Diary* and it was enthusiastically taken up by Gill in his characteristic manner. The first number which fell into his hands was that for 1830, and this contained problems for solution, of which Gill answered apparently a majority. For the issue of the year 1831, his answers were sent in too late for publication; but the *Mathematical Diary* for 1831 received one problem not too late for publication,—No. XXI, by "C. Gill, New York," freshly proposed (p. 193). The number for 1832 contained answers by "Mr. C. Gill, teacher of mathematics, Sawpits, N. Y.," who solved all but No. XXII and proposed two new problems, No. VIII and No. XVI.

Williams on the War Path.

In 1832 occurred the episode referred to in Bradley's *Memoirs of Strong,* already mentioned, and also by Cajori, in his *Teaching and History of Mathematics in the United States* (note to p. 96) of a general challenge addressed to the mathematicians of the United States by one John D. Williams. This man advertised in the *New York Commercial Advertiser* of November 30, 1831, "Algebraic questions by Diophantus, which he requests the mathematicians of the United States to solve—Dr. Bowditch, Professor Strong, E. Nulty, Dr. Anderson, C. Gill, and J. Cunliff being excepted. "Then follow fourteen questions (see Appendix B). Considering the little knowledge of Gill which was to be found in America within a year of his arrival, the fact that the *Mathematical Diary* for 1831 could contain enough of his work to make him so well known is very striking. Six months later, June 1, 1832, Mr. Williams inserted a renewal of his challenge which may also be found in Appendix B. It will be seen that Mr. Williams changed his list of excepted names. The names of Gill and Professor Henry J. Anderson, of Columbia College, are returned to the list of challenged. There may have been some animus against Gill to account for his transfer from the rank of Sheep to that of Goat, or perhaps it was made in good faith, based on a careful study of previous numbers of the *Mathematical Diary.*

The only reason for a contrary suspicion is based on the unfriendly reference to him in a footnote to the advertisement given in Appendix B. It will be seen by the Appendix that "a certain teacher in this city who goes by the nickname of Professor has offered to solve the 9th question for $5.00." Williams says enough to point out that Gill is the culprit, in saying "a certain teacher in this city." His statement implies that the teacher has been in correspondence with him. He says that he has not received any correct solution to all the questions, showing that some one—Gill, to wit—had endeavored to solve some of them. He gives Gill's address as Sawpitts Academy, in the obsolete spelling of the previous century, though Gill himself must have given it, as he did to the forthcoming *Diary* of 1832, in the more modern form. There were only two others in the list not known to be professors, except Mr. Floy and Samuel Ward, 3d. Ward is separately called by Williams "this gentleman," the last on the list; and I have personal knowledge that Floy was a man of brilliant parts, a friend of my own family, who might conceivably be a teacher, but if so he would have given his address. Both men were, like the others in the list, known to the editor of the *Mathematical Diary*. In short, it would seem that Williams made a personal attack on Gill, spiteful and vicious, of which any reader can judge as well as I. Gill had obviously proposed to take $5.00 for one solution, and had proposed it probably to get money from a supposedly rich advertiser. Williams replied by way of the newspapers, offering him $20.00 to prove whether a solution was possible or impossible. Perhaps Gill was really called Professor before his time by some of his students. Half the actuaries in the Society have suffered by the misuse of this same title.

During his stay at the Sawpits Academy Gill's reputation was gradually built up by his work in the *Mathematical Diary,* until it stood in the first rank in the year 1832. Probably early in the next year there happened one of those fortunate occurrences which sometimes color a whole lifetime. Gill saw an advertisement which had been inserted by Dr. Muhlenberg, showing the need of a mathematical teacher at Flushing, Long Island. Gill regarded this advertisement as something precious, and kept about him a clipping from the paper containing it for more than twenty years until his death, when, says Dr. Muhlenberg, it was found in his pocket.

St. Paul's College, Flushing.

Some account of his life at Flushing has been in the previous paper, to which the reader is referred. The time passed happily and peacefully, though at least one occasion was found by Muhlenberg for fearing "that he supposed that I wished him to dispense with his services." Muhlenberg says, in substance, that Gill was hardly patient enough with stupid boys, having heard from him that he had, in his own boyhood, "read the first six books of Euclid in one day." "He gradually seemed, however, to accommodate himself to the average mind of the young, and made good scholars." "He would never allow that he had made more than one mathematician," but to this one rare bird, whose name is not given, at Gill's request Muhlenberg awarded a gold medal, the only medal ever given in the college. We can imagine the enthusiasm with which the young man was taken up by Gill, as a man after his own heart, and the joy with which he treated the news of his success subsequently at the Ecole Polytechnique in Paris, where he achieved rapidly a "distinguished reputation," a reputation which showed the St. Paul's medal, granted obviously only after much insistence by Gill, to have been wisely bestowed after all. Dr. Muhlenberg says: "While at Flushing Mr. Gill made a translation of the great work of La Grange, which still remains in manuscript." Which great work is meant I do not know. No trace of it is left among the papers that are preserved in the office of the Mutual Life. Dr. Muhlenberg stated that Gill "became a communicant of the Episcopal Church some three years after he formed his connection with me." This somewhat indefinite statement I take to mean that Gill was not confirmed at that time, but was admitted to communion as a member of the Church of England by Dr. Muhlenberg, as president of the college, being himself a clergyman.

The *Mathematical Diary* not appearing after 1832, its place was filled by the appearance of the *Mathematical Miscellany* in the spring of 1836 under the editorship of Charles Gill. I broke off my account of Gill's opening paragraph in my first paper too abruptly, from the irksomeness of copying, and now give it as Appendix C.

There is lacking, in the copy before me, a small fragment, which had been printed apparently as a loose piece of paper, not paged, in front of the third number of the *Miscellany* and bound in with at least one copy, which is preserved in the Library of Columbia

University. This fragment has been called to my attention by Mr. Joffe, and for personal reasons I venture to mention it to the Society. It gives for the information of the readers of the third number, which was printed in the spring of 1837, several names of persons willing to make an annual contribution for a stated amount, including Professors Penney and Strong pledging $15 each, and several other names, six in all, of which I select those of Professors Caldwell and McClintock, Dickinson College, Carlisle, Pa., $5.00.*

Gill brought out his *Miscellany* by semi-annual numbers regularly for four years, ending with 1839. In the first number he began a treatise on *Spherical Geometry,* which was completed in the third number. His *Hints to Young Students,* having a purely pedagogic character, which are given clearly and must have been found effective, were continued through several numbers.

In the fourth number began a series of articles on the Diophantine Analysis, by William Lenhart, which he named *Diophantine Speculations,* which would be interesting reading at the present day. The sixth number had two articles by distinguished writers, one following the other, respectively by Professors Strong and Peirce, one on the orthographic projection of the circle, and the other, somewhat longer, giving an account of a recent English work on the Integral Calculus.

The last number (the eighth) for November 1, 1839, contains an article by Δ, on the application of Sturm's Theorem, which happens to be personally interesting to me. This gives for the fourth and fifth degrees respectively, the real roots if they exist, and the rules which enable one to determine them. The details are given, and

* Merritt Caldwell and John McClintock, who held the scientific chairs at that time: John McClintock, my father, was Professor of Mathematics, just entering on life, having been graduated in June, 1835, from the University of Pennsylvania at the age of 20. As the youngest member of the faculty, the smallness of the amount contributed by him, about $2.50, may be excused. Some early manuscript books and boyish exercises are all the relics of that time which I have of him. Among them I find the advertisement printed on the last page of a text-book that he at the age of seven had attained proficiency by the use of this particular grammar. From some of his students I have gathered that he was alert, cautious, and stimulating in his classes. He was a born orator. I knew him later as a theologian, a scholar of many branches, and a voluminous writer; his books including a series of Latin and Greek text-books. He had the genius which consists in "an infinite capacity for taking pains," being laborious to the last degree, literally killing himself with work at the age of 55.

coincide, in each case, with the results (special cases) of my own paper on the same subject.*

It has been previously stated that the last number closes abruptly without explanation or warning, and the *Mathematical Miscellany* came to an end with Gill's four years' editorial work, at the same time. In this same year, 1839, Gill re-appeared in the *Ladies' Diary,* writing from and dating his article from St. Paul's College, Flushing. This article is "On the General Equation of the Fifth Degree." It appears to be really a copy of the article by Δ, above referred to, and, if so, Δ was Gill himself. This article is given in Appendix D.

In 1842, Gill's son, William C. Gill, was born.

The preface of Gill's first book, the only one which he is known to have written, entitled "*Application of the Angular Analysis to the Solution of Indeterminate Problems of the Second Degree*" was dated March 29, 1848. This was published in 1848, by John Wiley, New York, and by Wiley and Putnam, London, and contained 91 pages. The publishers stated recently that "the book is extremely scarce and most of the impression was destroyed." The chief object of the preface is to acquaint the reader that the applications of his new theory are mainly devoted to the Diophantine Analysis. Obviously, the questions so boastfully advertised by Mr. Williams had not been forgotten. I regret that this little book is too long for insertion in an Appendix.

Close of the Second Paper.

I am obliged to close the second paper abruptly, as I did the first. The second is already about twice as long as the first, including some parts which are most conveniently printed as Appendices. The third paper, partly written, will be contributed later. The Society is long-suffering, as my experience of it has proved, but I do not think that even the necessary appendices will be found too trying, considering the honor which the Actuarial Society owes to the memory of the first actuary in America. Gill was the first actuary of the Mutual Benefit as well as of the Mutual Life, as has

*"A Method for Calculating simultaneously all the Roots of an Equation." This is summarized in Merriman's *Solution of Equations,* pp. 29–31, published by John Wiley & Sons, New York, giving me full credit, citing Vol. XVII of the *American Journal of Mathematics,* and stating that, "by this method all the roots, whether real or imaginary, can be computed from a single series."

been stated in various publications. The two periods of service were, in both cases, highly appreciated; the last, short as his life was, extending to six years; and both left important traces, little as was known, until lately, about him.

APPENDIX A.

Extracts from the "Ladies' Diary: or, Complete Almanack" bearing upon Charles Gill's Contributions

No. 120 (1823), *pp. 47–48.*

New Mathematical Questions.

I. Quest. (1404); by Mr. Joseph Gritton, London.

The diameter of the base of a right cone is equal to its slant height; and its solid and superficial contents are numerically equal. Required the dimensions and content.

II. Quest. (1405); by Mr. Thomas Baker, Elm Park, Durham.

$AHCB$ is a given segment of a circle, AB the chord, H the middle of the arc, and C any point in it: join AC, CB, CH; then $AC \cdot CB + CH^2 =$ a constant quantity. Required the demonstration.

III. Quest. (1406); by Mr. Tho. Todd, Northwich.

Given the sum of the sides which include the right angle, when the square of half the greater of those sides is equal to their rectangle; to construct, geometrically, the right angled plane triangle.

IV. Quest. (1407); by Mr. Isaac Newton, Lynn Regis.

Let the variable right line DEB revolve upon B as a centre in the same plane; and let it be always so divided at a point E as to make $DE^2 + EB^2 =$ a given square. Required the loci of the points D and E.

V. Quest. (1408); by the Rev. John Sampson, Kendal.

If to the three angles of any plane triangle, lines be drawn from these three points, viz. the centre of the circumscribing circle, the centre of the inscribed circle, and the intersection of the perpendiculars (from the three angles upon the opposite sides); the line drawn to either angle from the centre of the inscribed circle, always bisects the angle made by the lines drawn thence from the other two points. Required a demonstration.

VI. Quest. (1409); by Mr. W. Burdon, York.

The number of vibrations made by a globe when suspended by a point on its surface, is to the number made in the same time when suspended by a string twelve inches in length, as seven to six. Required the diameter of the globe.

VII. Quest. (1410); by Mr. John Baines, Bawtry.

Required the angle which the line drawn through the moon's horns will make with the vertical plane; also the point in which that line continued will cut the horizon, on Aug. 11th next, at 6 h. P. M. at Greenwich; by a general method applicable to any other time and place.

††† A similar question appeared in the Diary some years ago; but the problem is thought susceptible of a better solution.

VIII. Quest. (1411); by Mr. J. Brown, Epping.

In a right angled triangle ABC, there is given the hypothenuse $AC = a$, to find its area when $AB^n \times BC^{n-1}$ is a maximum.

IX. Quest. (1412); by Mr. R. Maffett, Plymouth.

Let C, C' be two fixed points 50 inches asunder; and let CP, $C'P'$, be two inflexible bars each 24 inches long, moving freely about the respective centres C, C': let $PP' = 14$ inches be an inflexible bar playing freely upon pivots or joints at P, P'. It is required to investigate the nature and quadrature of the curve MRS, described by M, the middle point of PP', while the three connected rods CP, PP', $P'C'$, move freely about the centres C, C'.

N. B. The system here described is obviously analogous to "the parallel motion" in Mr. Watt's steam engine.

X. Quest. (1413); by Mr. W. D. Rangeley, Stone.

Two points are given by position, also a plane inclined to the horizon in a given angle. Each end of a flexible string being fastened at one of the given points, and a ball of a given diameter being hung on the string by means of a very small hook; it was observed that when the ball was at rest it just touched the plane. It is required to determine, geometrically, the point of contact.

XI. Quest. (1414); by Mr. Matthew Neil, Anderton, near Glasgow.

Let AB, BC, CD, be arcs of a circle of given diameter, and P any point within that circle, to which right lines AP, BP, CP, DP, are drawn from the extremities of the several arcs. Now the arcs AB, CD, are known, as well as the angles APB, BPC, CPD; whence the middle arc BC is required.

XII. Quest. (1415); by Mr. W. Wiseman, Hull.

Let M apparent, and M' true altitude of the moon; S and S' the apparent and true altitudes of the sun or a star; D and D' the apparent and true distances of the moon and sun, or star: then the versin $D' =$ versin $(M' \backsim S') +$ cos. M' cos. S'

$$\frac{\text{versin } D - \text{versin } (M \backsim S)}{\text{cos. } M \text{ cos. } S}.$$ Required a demonstration.

XIII. Quest. (1416); by the Rev. John Sampson, Kendal.

It is required to find the sides of a scalene triangle ABC in integers, such that the area may be an integer, and the three lines

AO, BO, CO, may be likewise integers, *O* being the centre of the inscribed circle.

XIV. Quest. (1417); by James Thomson, A.M. Professor of Math. in the College of Belfast, and Author of a Treatise on Arithmetic.

Prove that of the two infinite serieses

$$1-\cos \tfrac{1}{3}\phi+3^2(1-\cos \tfrac{1}{9}\phi)+3^4(1-\cos \tfrac{1}{27}\phi)+3^6(1-\cos \tfrac{1}{81}\phi)+\text{etc.}$$

and

$$1-\cos^3 \tfrac{1}{3}\phi+3^2(1-\cos^3 \tfrac{1}{9}\phi)+3^4(1-\cos^3 \tfrac{1}{27}\phi)+3^6(1-\cos^3 \tfrac{1}{81}\phi)+\text{etc.}$$

three times the first exceeds the second by $\frac{1}{8}\phi^2-\frac{1}{2}\sin^2\frac{1}{2}\phi$.

XV. Or Prize Question (1418); by Investigator.

[Whoever answers it before Feb. 1, has two chances for ten Diaries.]

If the fraction $\frac{1}{23}$ be converted into a decimal, it will be a repetend, or circulate; viz.

$$.\dot{0}4347826086$$
$$95652173913\dot{}$$

in which these properties obtain.

1. The number of places in the circulating period is $23-1$; and whenever the denominator is a prime number n (except 2 or 5) the number of places in the circulating period is always either $n-1$, or $(n-1)\div m$, m being an integer less than n.

2. The first and the last half of the period consist of figures, which, added term by term, constantly make 9: as, $0+9$, $4+5$, $3+6$, etc.

3. The same circulating period includes all the repetends respectively equivalent to $\frac{2}{23}$, $\frac{3}{23}$, $\frac{4}{23}$, etc., each commencing at an easily assignable place: thus, $\frac{3}{23}=.13043478$, etc.

Required a demonstration of these properties applicable to all fractions whose denominator is a prime number, greater than 5.

No. 121 (1824), *p. 33.*

Mathematical Questions Answered

[List of Mathematical Contributors, and of Questions ans. by each.]

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C. Gill, Ganton, near Scarbro', Qu. 1 to 8, 10, 13.

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p. 34.

*** From this valuable collection of solutions, the Editor has, with much impartial solicitude, selected the following; with a constant regard to the interests of science, the encouragement of his young correspondents, and the reputation of all.

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p. 43.

XIII. Question, ans. by Mr. C. Gill, Ganton, near Scarborough.

Let ABC be the triangle, and O the centre of its inscribed circle. Put $AO=a$, $BO=b$, $CO=c$, and $DO=EO=FO=$ rad. of inscribed circle $=r$. Then, as AO, BO, CO, bisect the angles A, B, C, respectively, we shall have $BD=BF=\sqrt{b^2-r^2}$; $AF=AE=\sqrt{a^2-r^2}$; $CD=CE=\sqrt{c^2-r^2}$. Hence $AB=\sqrt{b^2-r^2}+\sqrt{a^2-r^2}$, $BC=\sqrt{b^2-r^2}+\sqrt{c^2-r^2}$, and $AC=\sqrt{a^2-r^2}+\sqrt{c^2-r^2}$. Now these sides will evidently be whole numbers, if (a^2-r^2), (b^2-r^2), and (c^2-r^2) are integral squares. Assume these $=m^2$, n^2, and s^2, respectively. Then $a=\sqrt{m^2+r^2}$, $b=\sqrt{n^2+r^2}$, and $c=\sqrt{s^2+r^2}$: and, if we take $r^2=p^2+2pm$, $=q^2+2qn=t^2+2ts$, we shall get $a=m+p$, $b=n+q$, $c=s+t$, and the area $=r(m+n+s)=wh$. From the preceding three equations, we deduce $m=(r^2-p^2)/2p$, $n=(r^2-q^2)/2q$, and $s=(r^2-t^2)/2t$; where we have only to assume such values for r, p, q, and t, as shall make m, n, and s, whole numbers.

Take, for example, $r=12$, $p=2$, $q=4$, $t=6$: then $m=35$, $n=16$, $s=9$. $AO=37$, $BO=20$, $CO=15$, $AB=51$, $BC=25$, $AC=44$, and the area $=720$.

The following are selected from the numerical results of different investigations:

. . . . . .

The proposer, $DO=165$, $OA=325$, $OB=429$, $OC=275$, sides 676, 616, 500.

. . . . . .

In all these we have omitted the areas; the smallest next to Mr. Gill's is Mr. Brown's, viz. 15,840.

pp. 46–48.

New Mathematical Questions.

I. Quest. (1419); by Mr. Henry Ward, Stafford.

There are three numbers, the difference of whose differences is 5, their sum 14, and the product of the first and third multiplied into the cube of the second is 270. What are the numbers?

II. Quest. (1420); by Mr. Joseph Waters, Newark-upon-Trent.

Required a general theorem, by means of which any natural number may be divided any number of ways into three rational cubes.

N. B. This gentleman is about to publish by subscription an extensive work on the Diophantine Analysis.

III. Quest. (1421); by Mr. R. Oats, St. Just, in Penwith.

If a quantity of water equal to a cubic mile were poured on an extended plane in contact with the earth at its middle point, what would be the area of the plane (supposed a circle) that the water would cover, and the greatest depth of water on it, supposing the earth's radius 4000 miles?

IV. Quest. (1422); by Investigator.

Find 10 numbers increasing successively by 1, in arithmetical progression, such that not one of them shall be a prime, by a general method applicable to any other case.

V. Quest. (1423); by Mr. G. Ellis, Acomb, near York.

The solidity of a cone is 301.5936 inches; and, it is known, that its centre of gravity is one inch farther from the vertex of the cone than the centre of the greatest sphere that can be inscribed in it. Required the dimensions of both the cone and the sphere.

VI. Quest. (1424); by Mr. W. Rutherford, Woodburn-school.

Let AB be the diameter of a given semicircle ACB, and CD any right ordinate on which is described the equilateral triangle CDP. Required the quadrature of the curve, which is the locus of the point P.

VII. Quest. (1425); by Mr. John Baines, Bawtry.

Let ABC be a triangle right angled at B, whose base AB is given; and, in the variable hypothenuse, AC, take AP such that $AP.BC$ may be always $= AB^2$. Find the equation and area of the curve which is the locus of P.

VIII. Quest. (1426); by Mr. Isaac Newton, Lynn Regis.

If from a given point P, either within or out of a given circle, a right line PC be drawn to the circumference, and PC be divided at E in a given ratio, the locus of E will be a given circle. Required a demonstration.

IX. Quest. (1427); by Pappus.

Draw any line BC meeting the side FA (produced) of the triangle ACF, and let the triangle ACF revolve about the line BC. Then is the solid generated by the triangle ACF, equal to a cone whose alt. is the perp. CE, and its base a circle equal to the superficies described by AF during the rotation. Required a proof.

X. Quest. (1428); by Mr. W. Mawson, Witton-le-Wear.

Demonstrate that an attractive force, which varies in the inverse duplicate ratio of the distance, cannot be constituted of several forces, each of which does not vary in that ratio.

XI. Quest. (1429); by Mr. W. Godward, Middlewich.

Of all cones of the same capacity, it is required to determine the dimensions of that along the slant side of which a heavy body will descend in the least time possible.

XII. Quest. (1430); by Mr. Jonathan Horner, Knaresborough.
Required the base diameter, and the altitude, of a paraboloid, when its solidity is 324, and the time of vibration a minimum, the point of suspension being the vertex.

XIII. Quest. (1431); by Mr. Thomas Squire, Epping.
On the 1st of December, whilst a lunarian astronomer was watching the progress of a solar eclipse, he observed a projection on the dark body of the earth, and which impeded the evanescent ray of the sun. Admitting this to have been the peak of Teneriffe, N. lat. 28° 17′ W. long. 16° 40′, and that the situation of the observer was exactly on the first point of ingress of the total shade; what would be the *time,* and on what part of the moon's periphery would the said eclipse be seen to begin at Greenwich, supposing the sun's declination to be 21° 52′ 20″ S, and the ☽'s semidiameter 16′ 13″?

XIV. Quest. (1432); by Mr. John Woodrow, Hingham, Norfolk.
A uniform lamina of matter in the form of a given triangle is suspended by a fine string, without weight or inertia, attached to one of its angular points, and made to vibrate edgeways. It is required to find the time of the small oscillations, the length of the string being such that a small variation therein may cause no sensible variation in the time of vibration.

XV. Or Prize Question (1433); by the Editor.
[Whoever answers it before Feb. 1, has two chances for ten Diaries.]
In the appendix to Mr. Cavendish's paper on the mean density of the earth, (Phil. Transac. vol. 88,) when computing the attraction of the mahogany case used in his experiments, he finds the attraction of a rectangular plane, as *BCDE,* in direction *BC,* by a logarithmic expression, but the attraction in the direction *AC,* perpendicular to that plane by a tedious series. Now, if $AC=a$, $CD=b$, $CB=x$, the attraction in the first direction will be the correct fluent of $\frac{bx\dot{x}}{(a^2+x^2)\sqrt{a^2+b^2+x^2}}$; while that on A in the direction AC will be the correct fluent of $\frac{ab\dot{x}}{(a^2+x^2)\sqrt{a^2+b^2+x^2}}$. It is required to determine both of these in terms either of logarithms or of trigonometrical quantities, without series.

No. 122 (1825), p. 33.

Mathematical Questions Answered.

[List of Mathematical Contributors and of Questions ans. by each.]

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Charles Gill, Ganton, near Scarbro', Qu. 1 to 12, Pr.

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pp. 36–37.

(After giving Mr. N. J. Andrew's solution to Question IV., the following note is added:)

Mr. Cha. Gill, of Ganton, deduces this Theorem, viz.—If p be any prime number, and m the product, or any multiple of the product of all the primes under p; then will $m+2$, $m+3$, $m+4$, etc. to (at least) $p-2$ terms, but often to many more, be a series of consecutive numbers, not one of which will be a prime.

p. 41.

X. Question, ans. by Mr. W. S. B. Woolhouse (aged 14);

. . . . . .

Otherwise, by Professor Thomson, and Mr. Charles Gill.

Let D denote any fixed distance from the centre, and A a constant quantity; then ϕD, $\phi' D$, $\phi'' D$, etc. denoting any functions of D, except m/D^2, m'/D^2, etc. m, m', etc. being constant quantities, we are to prove that $\phi D + \phi' D + \phi'' D + \text{etc.} = A/D^2$, cannot be true when D varies. Now, let D become $\Delta + D$; then $\phi(\Delta + D)$, $\phi'(\Delta + D)$, etc. may be expressed by series, and we should have

$$\left.\begin{array}{l} a \;+b\;D + c\;D^2 + \text{etc.} \\ a' +b'D + c'D^2 + \text{etc.} \\ a'' +b''D + c''D^2 + \text{etc.} \\ \quad + \text{etc. etc. etc.} \end{array}\right\} = \frac{A}{(\Delta + D)^2},$$

where a, a', b, b', etc. are constant coefficients. Multiplying by $(\Delta + D)^2$ and contracting, we should have a series of the form $\alpha + \beta D + \gamma D^2 + \delta D^3 + \text{etc.} = A$, a constant quantity; which is impossible, if D vary.

APPENDIX B.

New York Commercial Advertiser.

Vol. XXXIV. New York, Wednesday Evening, November 30, 1831.

For the Commercial Advertiser.

Algebraic questions, by Diophantus, which he requests any mathematician in the United States to solve—Dr. Bowditch, Professor Strong, E. Nulty, Dr. Anderson, C. Gill, and J. Cunliff, being excepted.

1. Make $x^2 + y^2 = a^2 = z^2 + w^2 = \text{sq.}$, and $x^2 - w^2 = z^2 - y^2 = \text{sq.}$
   Ans. if $a = 7585$, then $x = 7400$, $y = 1665$, $z = 6273$, $w = 4264$.
2. Make $x^2 + y^2 = b^2 = z^2 + w^2$ and $x^2 - z^2 = w^2 - y^2 = \text{sq.}$
   Ans. if $b = 697$, then $x = 680$, $y = 153$, $z = 672$, and $w = 185$.
3. Make $x^2 \pm 23256x = \text{sq.}$, $x^2 \pm 75000x = \text{sq.}$, and $x^2 \pm 103224x = \text{sq.}$
   Ans. $x = 105625$.

4. Make $x^2 - 23256^2 = \square$, $x^2 - 75000^2 = \square$, $x^2 - 103224^2 = \square$.
Ans. $x = 105625$.

5. Make $x + y + z = a^3$, $x + y = b^3$, $x + z = c^3$, and $y + z = d^3$ in positive whole numbers.

6. Make $x^2 \pm 96x = \square$, $x^2 \pm 135x = \square$, and $x^2 \pm 154x = \square$.

7. Make $182^2x^2 \pm 182x = \square$, $560^2x^2 \pm 560x = \square$, and $630^2x^2 \pm 630x = \square$.

8. Make $136^2x^2 \pm 136x = \square$, $170^2x^2 \pm 170x = \square$, and $174^2x^2 \pm 174x = \square$.

9. Make $(m^2 + n^2)^2x^2 \pm (m^2 + n^2)x = \square$, $(m^2 - n^2)^2x^2 \pm (m^2 - n^2)x = \square$, and $4m^2n^2x^2 \pm 2mnx = \square$.

10. Make $\frac{xyz}{x+y+z} = \square$, $x^2 + \frac{xyz}{x+y+z} = \square$, $y^2 + \frac{xyz}{x+y+z} = \square$, and $z^2 + \frac{xyz}{x+y+z} = \square$, all in whole numbers.

11. Make $(m^2 + a^2)^2 \pm 4abm(m^2 + a^2)(m^2 - a^2) = \square$, and $(m^4 + 2m^2a^2 + a^4) \pm 4acm(m^2 + a^2)(m^2 - a^2) = \square$.

12. Make $x^2 + y^2 = \square$, $\frac{5}{4}(x^2 + y^2) =$ a cube, $xy = 2x^3$, $2(x + y) + \frac{xy}{x+y} = \square$, and $(x^4 + y^4)(x^2 + y^2) - (x^5 + y^5)\sqrt{(x^2 + y^3)} = \square$.

13. Make $x^2 \pm 97104 = \square$, $x^2 \pm 150000 = \square$, $x^2 \pm 173400 = \square$, and $x^2 \pm 180576 = \square$.

14. Find the least values of x, and y, in whole numbers that solve the equation $x^2 - 940751y^2 = 38$.

New York Commercial Advertiser.

Vol. XXXV. New York, Friday Evening, June 1, 1832.

Communication.

Messrs. Editors—It is this day six months since under the signature of *Diophantus,* I proposed through the medium of your paper, to the mathematicians of America, a collection of problems in Diofantine Analysis. No correct solutions having as yet been received to the whole of them; I take this opportunity to fulfill *my* pledge to furnish such, and enclosed they will come to your hands. I now desire to re-propose them for the ensuing six months; and shall except from my challenge the Hon. Nathaniel Bowdich, L.L.D. &c. &c. of Boston, Mass., Mr. Eugene Nulty, of Philadelphia, and Professor Theodore Strong, of Rutgers College, New Brunswick, N. J., *only.* The list of gentlemen challenged stands then, as follows:

Professor Robert Adrain, University of Pennsylvania.
Henry J. Anderson, Columbia College, N. York.
Benj. Peirce, Harv. Univ., Cam. Massachusetts.
Mr. J. Ingersoll Bowdich, Boston, Mass.
" Marcus Catline, Hamilton College, Clinton, N. York.

Mr. M. Floy, jun., New York.
" C. Gill, Sawpitts Academy, New York.
" L. L'Inconnew, Cincinnati, Ohio.
" Benjamin Hallowell, Alexandria, D. C.
" Charles Farquhar, Alexandria, Maryland.
" Samuel Ward,* 3d, New York.

It being presumed that there are none in the United States, with the exception of the above list would think of attempting their resolution, the questions proposed are as follows: (Here follow the same 14 questions as were published on November 30, 1831.)

* I beg leave to state that on the 2d day of March last, I received from this gentleman correct solutions to questions 1, 2, 3, 4, 5, 10, 13 and 14. This I suppose is about his *ne plus ultra*—beyond it I defy him to advance.

"Till riper age shall with maturer force burnish his mind."

☞A certain teacher in this city who goes by the nick-name of *Professor* has offered to solve the 9th question for $5.00; I now offer him $20.00 to prove either its *possibility* or *impossibility,* and shew all the conditions that can exist, and those that cannot.

QUERY.—Will Professor James Renwick favor me with an explanation of the last 20 lines, Book 1, page 8, of his excellent "Elements of Mechanics," just published.

Yours very respectfully,
JOHN D. WILLIAMS.

NEW YORK, June 1st, 1832.

APPENDIX C.

ADVERTISEMENT (BY GILL ANNOUNCING THE FIRST ISSUE OF THE *Miscellany*).

The first number of the *Mathematical Miscellany* is an experiment. It is presented to the public without the courtesy of a prospectus, in the belief that its character and claims to patronage will be better understood by giving a specimen, than a promise of what it will be.

The high estimation in which such works are held by the European mathematicians, and the fact that the great improvements in Analysis, and the vast variety of elegant problems scattered through the elementary works on science in present use, have generally first appeared in them, and are principally due to the discussions and investigations they are calculated to bring forward, are at least presumptive evidence that a miscellany of this kind might be made sufficiently interesting, were the talent of the country, of which there is certainly no want, concentrated in its aid.

The advantages of such a work, as a medium for valuable communications that might otherwise be lost to the public; as an index

to mark the taste in science, and the progress in discovery, of the day and of the country; and as a field where the aspirant to mathematical distinction may try his strength with those of established reputation, will be perceived at once by all who would think of patronizing this undertaking.

The Editor has the assurance of assistance from individuals whose names would be a sufficient guarantee for the respectability of the work; and if he succeeds in establishing it, he has no doubt of enlisting in its aid much of the mathematical talent of the United States. He begs leave to commend his undertaking, in particular to gentlemen of the mathematical chairs in our colleges, with the suggestion, whether it might not be made a useful auxiliary in cherishing a spirit of science in their classes. Should this suggestion meet their view, there will be formed a distinct department adapted to this purpose; and pains will be taken to make this part of the work interesting, for it will be peculiarly gratifying to the Editor, if he can supply the means in any degree of fostering the emulation of American youth in a study which is peculiarly adapted to the enquiring mind, and which is daily becoming of more practical importance to the country.

The *Mathematical Miscellany* will appear semi-annually on the first days of March and October; thus making the summer interval of seven months, and the winter one of five; a distinction which will be at once appreciated by the *student*. The price of each number will be 50 cents, and as it is not designed to secure any profit from the publication, the size of the work will be increased to whatever extent its sales will allow.

APPENDIX D.

From the Ladies' Diary for 1839.

ON THE GENERAL EQUATION OF THE FIFTH DEGREE.

By Mr. C. Gill,

Professor of Mathematics in St. Paul's College, Flushing, Long Island.

The Note appended to the 58th page of the last Diary will lead young mathematicians to entertain an opinion of Sturm's Theorem, by no means commensurate with its merits. If it could be applied to no other purpose than the solution of numerical equations, it has the merit of being infallible, as well as direct in its application. Even under the circumstances where other methods might be used to detect the nature of the roots with greater facility, a modification of this theorem may be used, which will require no greater labour, and which has still this advantage, that whereas even Mr. Horner's method may only "induce a suspicion," which it may require other

and distinct processes to confirm or to overthrow; by this method the actual difficulty is detected, and a recurrence to the correct and true process is at once easy and direct. But it is not alone in its application to numerical equations that Sturm's Theorem is valuable: it can be applied to cases where the coefficients are functions of one or more indeterminate constants, and thus be made subservient to the general purposes of analysis. Every analyst has felt how tantalizing it is when, in the course of his investigations, he arrives at a resulting equation of a degree greater than the third; and when, in most cases, he only wishes to know the nature of the roots, or the relation of the constants, which will render them of some assigned nature, he is, in fact, obliged to abandon the enquiry. A remarkable example of the kind of investigation I mean, when the resulting equation is of the third degree, may be seen at page 13, vol. i., of the "Mathematical Miscellany;" and if Sturm's Theorem can supply us with similar criteria for equations of higher degrees, it will fill a gap in the requirements of algebra that no other known method could do. I have applied this process to the general equation of the fourth degree, and the results will be published in No. 7 of the "Miscellany;"* and I propose to give here some of the more prominent results when it is applied to the general equation of the 5th degree, confident that when the readers of the Diary feel that this theorem may thus be made the instrument of analysis, they will hasten to modify and extend them.

Taking the Theorem of Sturm, as given in Professor Young's "Theory of Equations," and preparing the polynomials, by multiplying them by the square, or essentially positive denominators of their coefficients, we have, for the general equation of the fifth degree,

$$x^5 + ax^3 + bx^2 + cx + d = 0,$$

$$\begin{aligned}
X &= x^5 + ax^3 + bx^2 + cx + d,\\
X_1 &= 5x^4 + 3ax^2 + 2bx + c,\\
X_2 &= -2ax^3 - 3bx^2 - 4cx - 5d,\\
X_3 &= (40ac - 12a^3 - 45b^2)x^2 + (50ad - 8a^2b - 60bc)x - 4a^2c \\
&\qquad - 75bd\\
&= Ax^2 + Bx + C.\\
X_4 &= (4cA^2 + 2aB^2 - 2aAC - 3bAB)x + 5dA^2 + 2aBC - 3bAC\\
&= Dx + E,\\
X_5 &= BDE - CD^2 - AE^2.
\end{aligned}$$

Making thus, for the coefficients of the polynomial X_3,

$$\begin{aligned}
A &= 40ac - 12a^3 - 45b^2,\\
B &= 50ad - 8a^2b - 60bc,\\
C &= -4a^2c - 75bd;
\end{aligned}$$

* The Miscellany will generally be ready for distribution in London fifteen days after its publication in New York, that is, about the 15th of May and November of each year, leaving sufficient time for correspondents in Great Britain to prepare and transmit their communications.

and, for those of the polynomial X_4,

$$D = 4cA^2 + 2aB^2 - 2aAC - 3bAB,$$
$$E = 5dA^2 + 2aBC - 3bAC.$$

In calculating these coefficients, it must be recollected that if those of any one polynomial, as, for instance, A, B, C, have a common divisor, that divisor may be neglected. And it is also evident that, in cases where the ordinary methods might be advantageously employed, that is, where the real roots are widely separated, approximate values of these coefficients might be used; thus, if a contain two digits, A, B, C may contain seven digits under unfavorable circumstances; but if we divide them each by 10, or 10^2, or 10^3, &c., and neglect the remainders, the quotients may be used instead of the coefficients themselves, to calculate the succeeding coefficients. If any of the leading or terminating coefficients of the succeeding polynomials are found to approximate to zero, the true values must be used; for, under such circumstances, there will be roots either equal or having a very small difference.

Now, for $x = \infty$, these polynomials will become

$$X = +\infty, X_1 = +\infty, X_2 = -a\cdot\infty, X_3 = A\cdot\infty, X_4 = D\cdot\infty,$$
$$X_5 = BDE - CD^2 - AE^2.$$

Let i be the number of changes of sign in these quantities, and represent by m the number of real roots of the equation $X = 0$; then we shall have, as a known consequence of Sturm's Theorem,

$$m = 5 - 2i, \text{ or when } i > 2,\ m = 2i - 5.$$

Then we may form the following Table of conditions: when

			$X_5 > 0$		$X_5 < 0$	
$a > 0$	$A > 0$	$D > 0$	$i = 2$	$m = 1$	$= 3$	$m = 1,$
"	$A > 0$	$D < 0$	4	3	3	1,
"	$A < 0$	$D > 0$	2	1	3	1,
"	$A < 0$	$D < 0$	2	1	1	3,
$a < 0$	$A > 0$	$D > 0$	0	5	1	3,
"	$A > 0$	$D < 0$	2	1	1	3,
"	$A < 0$	$D > 0$	2	1	3	1,
"	$A < 0$	$D < 0$	2	1	1	3.

The second of these series of conditions are incompatible; since, if $a > 0$, $A > 0$,

$$40ac = A + 12a^3 + 45b^2, \text{ and } c > 0;$$
$$\therefore D = 4cA^2 + 2aB^2 - A(2aC + 3bB)$$
$$= 4cA^2 + 2aB^2 + 4cA(2a^3 + 45b^2) + 24a^2b^2A$$
$$> 0.$$

It follows that if $X_5 > 0$, there must be either five real roots, or only one; and there are five only when, at the same time, $a < 0$, $A > 0$, $D > 0$. When $X_5 < 0$, there may be one real root, or three real roots, but no more; and there will be three when, at the same time, either

$$1°,\ A < 0,\ D < 0,$$
$$\text{or } 2°,\ a < 0,\ A > 0.$$

To find the nature of these roots: if $x = 0$, the polynomials become

$$X = d,\ X_1 = c,\ X_2 = -5d,\ X_3 = C,\ X_4 = E,$$
$$X_5 = BDE - CD^2 - AE^2.$$

Let k be the number of changes of sign in these quantities, which is evidently not affected by any change in the sign of c, and m' the number of *positive* roots in the equation $X = 0$. Then will $m' = i \backsim k$. But since, when there is only one real root, its sign, from simple considerations, is opposite to that of d, we need only examine the two cases of five real roots, indicated by $X_5 > 0$, and $i = 0$; and three real roots, indicated by $X_5 < 0$ and $i = 1$. Then we may form the following Table of conditions:

when

			$X_5 > 0$ and $i = 0$		$X_5 < 0$ and $i = 1$	
$d > 0$	$C > 0$	$E > 0$	$k = 2$	$m' = 2$	$k = 3$	$m' = 2$,
"	$C > 0$	$E < 0$	4	4	3	2,
"	$C < 0$	$E > 0$	2	2	3	2,
"	$C < 0$	$E < 0$	2	2	1	0,
$d < 0$	$C > 0$	$E > 0$	1	1	2	1,
"	$C > 0$	$E < 0$	3	3	2	1,
"	$C < 0$	$E > 0$	3	3	4	3,
"	$C < 0$	$E < 0$	3	3	2	1.

Then, when there are five real roots, and $d > 0$,

4 of them are positive and 1 negative, when $C > 0$, $E < 0$;

2 of them are positive and 3 negative, in all other cases;

when there are five real roots, and $d < 0$,

1 of them is positive and 4 negative, when $C > 0$, $E > 0$;

3 of them are positive and 2 negative, in all other cases;

and there cannot be five positive roots, as we know from the circumstance that their sum $= 0$.

Also, when there are three real roots, and $d > 0$,

all of them are negative when $C < 0$, $E < 0$;

2 are positive and 1 negative, in all other cases;

when there are three real roots, and $d < 0$,

all of them are positive when $C < 0$, $E > 0$;

1 is positive and 2 negative, in all other cases.

We may state, moreover, from the mere inspection of the polynomials, that when $X_5 = 0$, there are two equal roots; that when $D = 0$ and $E = 0$, there are two pairs of equal roots; that when

$D=0$ and $E=0$, and at the same time $B^2=4AC$, there are three equal roots, &c.

EXAMPLE.

In the second example given by Mr. Davies, in the Prize Question of last Diary,

$$X=x^5-10x^3+6x+1=0;$$
$$\text{or } a=-10,\ b=0,\ c=6,\ d=1;$$
$$\therefore A=9600,\ B=-500,\ C=-2400;$$

or we may take $A=96,\ B=-5,\ C=-24.$

Then $$D=43651,\ E=10920,$$

or we may take to calculate X_5, $D=4$, $E=1$, and we find

$$X_5=272>0.$$

So that $a<0$, $A>0$, $D>0$, $X_5>0$, and there are five real roots. Also, since $d>0$, $C<0$, $E>0$, two of these roots are positive and three negative. By writing down the polynomials whose coefficients are here calculated, we shall immediately find the initial figures of these several roots. There is one between 0 and 1, one between 2 and 3, two between 0 and -1, whose initial figures are —, 1 and —, 6; and one between -2 and -3. Mr. Horner's method may now be employed to approximate to these roots.

*** The Editor has before mentioned, and again begs leave to recommend to English readers Mr. Spiller's translation of Sturm's valuable disquisition.

A Theory of Sub-standard Lives.

(Second Paper.)

BY

ALBERT W. WHITNEY.

In a paper which was read before this society a year ago I outlined a method for the actuarial treatment of the problem of sub-standard lives. In the present paper I want to add certain matter that has come up, first in the way of simplification and extension, and second, in the attempt to make a practical application of the plan.

The method depends upon the mathematical properties of certain functions which arise when actuarial quantities are expressed by means of the Makeham law of mortality, and particularly in certain properties of invariancy which these functions possess.

The Makeham formula for the force of mortality is

$$\mu_x = \alpha + \beta e^{\gamma x}. \tag{1}$$

α is that part of the force of mortality which is independent of age, and, presumably, will to a considerable extent be affected by environment and occupation. β is the value at age 0 of that part of the force of mortality* which is cumulative with age. This may probably represent the effect of an impaired constitution or other physical defect. γ is the rate per unit at which the growing part of the force of mortality is increasing. This may perhaps be influenced by habits, but it doubtless depends still more fundamentally upon the wearing qualities of the body, the rate, for instance, at which the arteries wear out.

The three parameters, α, β, γ, are connected with the ordinary Makeham parameters, s, g, c, by the relations:

$$\begin{aligned} \alpha &= -\log_e s, \\ \beta &= -\log_e g \log_e c, \\ \gamma &= \log_e c. \end{aligned} \tag{2}$$

I assume that we are provided with a standard table of mortality and that its Makeham constants, α, β, γ (or s, g, c), have been

* In this paper β has the same meaning as β_0 in the previous paper.

determined. I assume also that we are dealing with some particular type of sub-standard life whose mortality can be sufficiently well described by the Makeham formula but instead of the standard constants, α, β, γ, we have α', β', γ', as the values of the parameters.

Now it is a fact, I believe, that the constant, c, and hence γ, has been found to be very nearly the same for all mortality tables, and *a priori* it would seem to be reasonable to suppose that even among different circumstances the natural tendencies toward physical disintegration should be about the same.

For the sake of generality in the paper referred to I treated the sub-standard life $(\alpha', \beta', \gamma')$. As the aim of the present paper is to bring the treatment down to what is immediately practical, I shall instead consider the life $(\alpha', \beta', \gamma)$, that is I shall assume the constant, γ, to be normal but the α' and β' to be subnormal. This will be justified in the sequel, even though the considerations already referred to regarding the constancy of γ may not be sufficiently convincing in themselves.

The use of the standard γ greatly simplifies the work, and it can be carried through without the use of the integrals that were used in the previous paper. I will briefly repeat the argument for this simpler case, referring the reader, however, to the previous paper for a fuller explanation of some matters.

From the expression for μ, namely, $\mu_x = \alpha + \beta e^{\gamma x}$, and from the relation, $\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}$, we obtain by integration the equation:

$$\log_e {}_np_x = -\alpha n - \frac{\beta e^{\gamma x}}{\gamma}(e^{\gamma n} - 1). \tag{3}$$

In terms of the force of interest, v may be written $v = e^{-\delta}$. We have then:

$$\log_e v^n_n p_x = -(\alpha + \delta)n - \frac{\beta e^{\gamma x}}{\gamma}(e^{\gamma n} - 1). \tag{4}$$

We will now abbreviate as follows:

$$\frac{\alpha + \delta}{\gamma} \equiv h_i, \quad \frac{\beta e^{\gamma x}}{\gamma} \equiv b_x, \tag{5}$$

the subscripts indicating the fact that h is a function of i and that b is a function of x. Making these substitutions we have:

$$v^n_n p_x = e^{-h_i \gamma n} e^{-b_x(e^{\gamma n} - 1)}. \tag{6}$$

Let us now consider the sub-standard life which we may symbolize (α', β', x, i), meaning thereby that it is characterized by the Makeham parameters, α', β', that it is of age x and that the rate of interest that we shall employ is i. Let us also consider a standard life (α, β, x', i'), characterized by the standard Makeham parameters, α and β, but of a different age x', and let us furthermore employ a different rate of interest i'; x' and i' will be determined later.

Let us now form the function ${}_{n}v^{n}p_{x}$ for both these lives, indicating the use of the sub-standard Makeham parameters, α' and β', by primes upon the functions; h'_i, for instance, represents $\frac{\alpha' + \delta}{\gamma}$ and $b'_x = \frac{\beta' e^{\gamma x}}{\gamma}$, while the use of the age x' and the rate of interest i' will be indicated by primes upon x, i, v and δ.

We have then for the life (α', β', x, i),

$$[{}_{n}v^{n}p_{x}]' = e^{-h'_i \gamma n} e^{-b'_x (e^{\gamma n} - 1)},$$

and for the life (α, β, x', i'),

$$[{}_{n}v'^{n}p_{x'}] = e^{-h_{i'} \gamma n} e^{-b_{x'} (e^{\gamma n} - 1)}.$$

Now we will suppose that not only α and β but also α' and β' are definitely given, and, furthermore, that x' and i' are so determined as to make

$$h'_i = h_{i'},$$

and

$$b'_x = b_{x'};$$

that is

$$\frac{\alpha' + \delta}{\gamma} = \frac{\alpha + \delta'}{\gamma},$$

and

$$\frac{\beta' e^{\gamma x}}{\gamma} = \frac{\beta e^{\gamma x'}}{\gamma}. \tag{7}$$

These relations, when solved for x' and i', give

$$x' - x = \frac{\log \frac{\beta'}{\beta}}{\log c}, \tag{8}$$

(since $e^{\gamma} = c$), and $\alpha' - \alpha = \delta' - \delta$. This latter equation can,

however, be expressed in terms of i' and i in the form:

$$1 + i' = (1 + i)s^{1-\frac{\alpha'}{\alpha}}. \tag{9}$$

It will now be observed that when x' and i' are so determined, $[v_n^n p_x]' = [v_n'^n p_{x'}]$, that is, the function $v_n^n p_x$ is invariant under a transformation which replaces α' by α, β' by β, x by x', and i by i', where x' and i' are related to x and i by equations (8) and (9).

This means in words that it is possible to find a standard life of a different age and to find a different rate of interest for which the function, $v_n^n p_x$, will have the same value as for the sub-standard life. Whatever invariant property is possessed by $v_n^n p_x$ will obviously be possessed also by $\sum_n v_n^n p_x$ between any limits, and so we have the theorem that it is possible so to choose the age and rate of interest for a standard life that not only the value of a pure endowment but also the annuity values are the same as those for a given sub-standard life; furthermore this marking up of the age and the rate of interest is the same for all ages.

This means in practical application that, if standard tables for various rates of interest are at hand, it will be possible by the use of a single set of such tables to find the values of a pure endowment and annuities for any sub-standard life of the character that we have assumed, that is, in which γ is normal and obeying the Makeham law.

As an example we may take the case of a life for which the mortality is assumed for all ages to be fifty per cent. greater than normal, that is $\mu_x' = \frac{3}{2}\mu_x$ for all values of x. This gives $\alpha' = \frac{3}{2}\alpha$ and $\beta' = \frac{3}{2}\beta$. Substituting these values in equations (8) and (9), using 3 per cent. interest and Mr. Hunter's Makeham constants for the American Experience Table, we obtain the facts, first, that the age must be marked up 3.845 years and, second, that the rate of interest must be taken as 3.392 per cent. instead of 3 per cent.

While it is true that the annuity values are invariant under the given transformation, it does not follow that all actuarial functions are so. Take the premium for an ordinary life policy, for instance; we have:

$$P_{(\alpha', \beta', x, i)} = -d + \frac{1}{1 + a_{(\alpha', \beta', x, i)}}.$$

This, because of the invariancy of a_x, is equal to

$$-d + \frac{1}{1 + a_{(\alpha, \beta, x', i')}}.$$

But if we had made the transformation upon P we should have obtained

$$P_{(\alpha, \beta, x', i')} = -d' + \frac{1}{1 + a_{(\alpha, \beta, x', i')}},$$

which is not the same as $P_{(\alpha', \beta', x, i)}$; that is P in itself is not invariant; it is evident however that $P_x + d$ is invariant; similarly A_x is not invariant, but $\frac{1 - A_x}{d}$ is invariant.

The procedure in these cases may best be illustrated by an example. In the case just given in which the mortality was fifty per cent. in excess of that given by the American Experience Table, we found that the annuity value for the sub-standard life, using 3 per cent. interest, was the same as the annuity value for a standard life in which the age was increased by 3.845 years and the interest rate was 3.392 per cent. instead of 3 per cent.

By interpolation in the annuity tables which were given in the other paper I find the value of such an annuity for, say, age 35 to be 16.97.

I could not similarly have obtained the value of A and P by going into A and P tables with the age 38.845 and the interest rate 3.392 as I did for the annuity value, because the A and the P are not invariant. Instead the invariant functions are $\frac{1 - A_x}{d}$ and $P_x + d$, these being equal respectively to $1 + a_x$ and to $\frac{1}{1 + a_x}$. From these values we may compute A_x and P_x, the difference between this way and the erroneous way lying in the fact that we use d in so doing instead of d'. As a matter of practice, having obtained the value of the annuity, we go with this value into a three per cent. conversion table to obtain A and P; the values so got are $A = .47660$ and $P = .02652$.

The question arises whether the commutation functions are invariant. By an examination of this question we shall exhibit the whole matter in its most practical light.

From equation (6), by letting $n = x$ and $x = 0$ we obtain:

$$D_x = v^x l_x = l_0 e^{-h_1 \gamma^x} e^{-b_x} e^{\beta/\gamma}. \qquad (10)$$

Now it is evident that this is not invariant, for it contains the

quantities β and x outside of the invariant functions h_i and b_x. Notice however how the factors containing β and x drop out when we form the combination

$$\frac{D_{x+n}}{D_x} = v_n^n p_x,$$

an invariant.

$$v_n^n p_x = \frac{D_{x+n}}{D_x} = \frac{l_0 e^{-h_i \gamma x} e^{-h_i \gamma n} e^{-b_{x+n}} e^{\beta/\gamma}}{l_0 e^{-h_i \gamma x} e^{-b_x} e^{\beta/\gamma}} = e^{-h_i \gamma n} e^{-b_x (e^{\gamma n} - 1)},$$

as before.

Let us now find the relation subsisting between $D_{(\alpha', \beta', x, i)}$ and $D_{(\alpha, \beta, x', i')}$:

$$D_{(\alpha', \beta', x, i)} = l_0 e^{-h'_i \gamma x} e^{-b'_x} e^{\beta'/\gamma},$$

$$D_{(\alpha, \beta, x', i')} = l_0 e^{-h_{i'} \gamma x'} e^{-b_{x'}} e^{\beta/\gamma}.$$

Remembering that $h'_i = h_{i'}$ and $b'_x = b_{x'}$ and $e^{\gamma(x'-x)} = \beta'/\beta$, we have:

$$D_{(\alpha', \beta', x, i)} = e^{h'_i \gamma (x'-x)} e^{\frac{\beta'-\beta}{\gamma}} D_{(\alpha, \beta, x', i')} = (\beta'/\beta)^{h'_i} e^{\frac{\beta'-\beta}{\gamma}} D_{(\alpha, \beta, x', i')}. \quad (11)$$

From this equation we see that the D's for the sub-standard life and the equivalent standard life differ only by the factor $(\beta'/\beta)^{h'_i} e^{\frac{\beta'-\beta}{\gamma}}$. This is independent of the age; hence, the N's differ by the same factor, that is:

$$N_{(\alpha', \beta', x, i)} = (\beta'/\beta)^{h'_i} e^{\frac{\beta'-\beta}{\gamma}} N_{(\alpha, \beta, x', i')}. \quad (12)$$

Hence any quotient of homogeneous functions of N and D of the same degree is invariant. This again exhibits the invariancy of the annuity values and the pure endowment.

This simple relation does not subsist between the C's, as might be inferred from the fact that A and P are not invariant. $C_x = vD_x - D_{x+1}$,

$$\therefore C_{(\alpha', \beta', x, i)} = vD_{(\alpha', \beta', x, i)} - D_{(\alpha', \beta', x+1, i)}$$
$$= (\beta'/\beta)^{h'_i} e^{\frac{\beta'-\beta}{\gamma}} \{vD_{(\alpha, \beta, x', i')} - D_{(\alpha, \beta, x'+1, i')}\} \quad (13)$$

Now the quantity inside the brackets is not $C_{(\alpha, \beta, x', i')}$ for it contains the explicit quantity v instead of v'.

It is, therefore, not possible to use the C_x columns directly but a new function, $vD_{(\alpha, \beta, x', i')} - D_{(\alpha, \beta, x'+1, i')}$, must be computed to take the place of $C_{(\alpha, \beta, x', i')}$ and differing from $C_{(\alpha, \beta, x', i')}$ by its use

of v instead of v'. This is very simple and just as easy to compute as C.

In general a single office will do a sub-standard business on only one rate of interest, for instance, say, 3 per cent. In that case it will be necessary, therefore, in forming this substitute function with any rate of interest i' to use $v = \frac{1}{1.03}$ as a factor instead of v'.

Suppose that we call this function $\underline{C}$, that is: (definition)

$$\underline{C}_{(\alpha, \beta, x', i')} = vD_{(\alpha, \beta, x', i')} - D_{(\alpha, \beta, x'+1, i')}. \tag{14}$$

Then we shall have:

$$C_{(\alpha', \beta', x, i)} = (\beta'/\beta)^{h'i} e^{\frac{\beta'-\beta}{\gamma}} \underline{C}_{(\alpha, \beta, x', i')}. \tag{15}$$

Let us define $\underline{M}$ as follows:

$$\underline{M}_{(\alpha, \beta, x', i')} = \sum_{n=0}^{\omega-x'} \underline{C}_{(\alpha, \beta, x'+n, i')}; \tag{16}$$

that is, there is the same relation between $\underline{M}$ and the $\underline{C}$'s as between M and the C's. Then

$$M_{(\alpha', \beta', x, i)} = (\beta'/\beta)^{h'i} e^{\frac{\beta'-\beta}{\gamma}} \underline{M}_{(\alpha, \beta, x', i')}. \tag{17}$$

It will, therefore, follow that any quotient of homogeneous functions of the same degree in D, N, C or M will be invariant, if in the transformation $\underline{C}$ is taken for C and $\underline{M}$ for M.

Since all final actuarial functions can be put in the form of quotients of homogeneous functions of the same degree, it follows that the commutation columns can be made use of for a sub-standard life if a simple change is made in the C, M (and R) functions.

I wonder if the great simplicity of this fact is fully realized. It means that any sub-standard life can be handled by means of a single set of Makehamized tables for a standard life, exactly the same as ordinary tables in the D's and N's and only slightly different in the C's and M's. I say a single set of tables but I mean to include in this the fact that the tables must be computed for a number of rates of interest.

The reserve for an ordinary life policy is equal to

$$\frac{a_x - a_{x+n}}{1 + a_x},$$

and hence is invariant, and hence can be obtained directly from ordinary reserve tables by changing the age and the rate of interest. The reserves on all types of policy are not invariant, however. To investigate this point for level premium policies we may examine the u_x and k_x functions used in computing reserves by the retrospective method.

$$u_x = \frac{(1+i)l_x}{l_{x+1}} = \frac{1}{vp_x}.$$

This is invariant.

$$k_x = \frac{d_x}{l_{x+1}} = \frac{v}{vp_x} - 1.$$

This is not invariant. It can, however, be readily tabulated.

We have the retrospective formula:

$$({}_{n-1}V_x + \pi_x)u_{x+n-1} - k_{x+n-1} = {}_nV_x$$

Now if $\pi_x u_{x+n-1} - k_{x+n-1}$ is invariant for all values of n then, by mathematical induction, the reserves must be invariant. But

$$\pi_x u_{x+n-1} - k_{x+n-1} = \frac{\pi_x - v}{vp_x} - 1.$$

A necessary and sufficient condition for this to be invariant is that $\pi_x + d$ shall be invariant.

It follows, therefore, that in those cases, and in only those cases, will the reserves be invariant in which $\pi_x + d$ is invariant.

$\pi_x + d$ will be found to be invariant for policies which terminate at the end of the year in which the last premium is paid, that is ordinary life policies and the usual term and endowment policies; for in these cases

$$(\pi_x + d)(\mathrm{a}_x - v^n_n p_x \mathrm{a}_{x+n}) = 1 \text{ or } 1 - v^n_n p_x.$$

Having now found the actuarial methods which are to be used for a sub-standard life with a given α' and β', we must now raise the question of how the α' and β' are to be determined. The simplest judgment upon the sub-standard character of a life consists in ascribing to the class a constant excess of mortality for all ages. This constant might be taken as either additive or multiplicative. If it is additive, it means that α alone, the environmental parameter, is increased, and that, therefore, the change is made

entirely by an increase in the rate of interest. This produces an increase in the annual premium that is not very far from the same for most ages. In some offices the method of marking up the premium by a constant additive amount is in use for taking account of adverse occupational and environmental conditions.

It is the practice in some offices to take account of the effect of an impaired constitution or other physical defect by marking up the mortality by a given percentage for all ages. This method in general is known to be incorrect. In the case of overweight, for instance, the increased mortality is mostly in the later years. Nevertheless, this is a convenient rough and ready method.

In some of the offices where this method has been most fully elaborated the q_x's have been computed for various percentages of excess mortality, and, from these, commutation columns have been formed in the usual way. This, of course, necessitates a complete set of tables for each percentage of excess mortality.

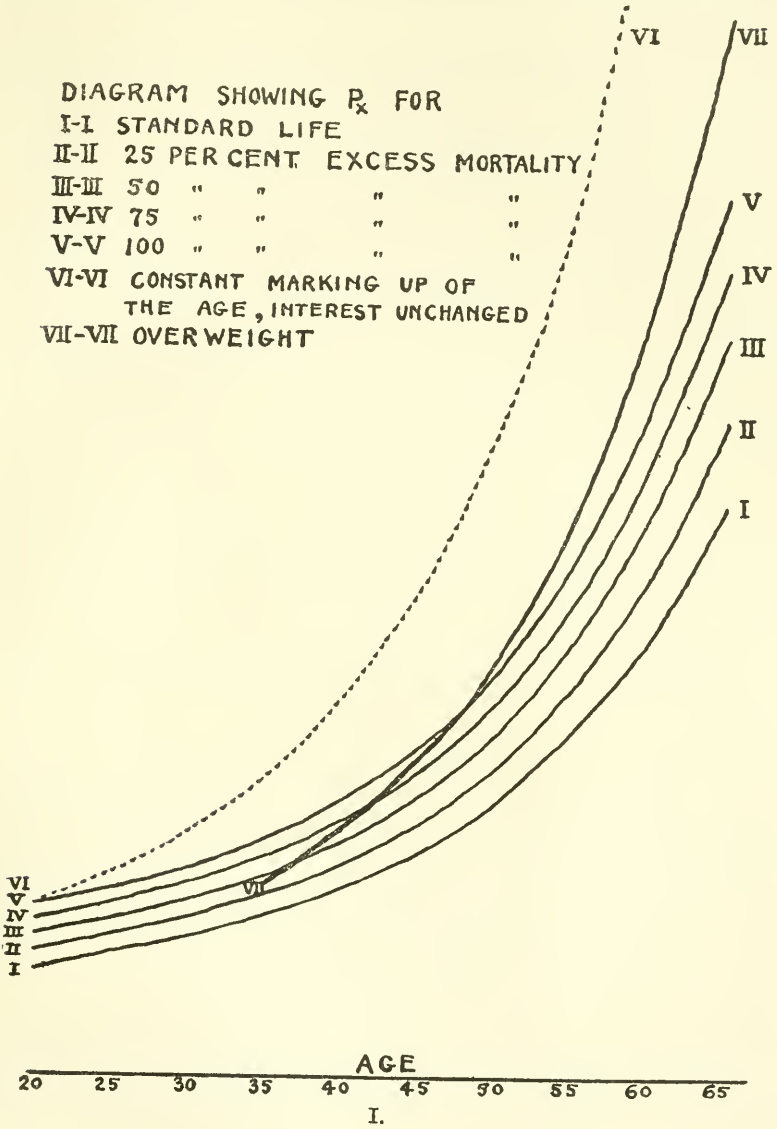
The method of this paper may be inferred from the case already given in which there is an excess mortality of fifty per cent.; the different cases will be characterized by different increments of age and interest, all however with reference to a single set of tables.

The attempt has been made in some offices to describe the increased premium in the case of a constant percentage of excess mortality by means of an increase in the age alone, but this gives results for the higher ages that are grossly exaggerated. The fact of the matter is that it is only by taking both age and rate of interest into account that the proper results can be obtained; all other methods are doomed to failure. An increase in the age alone would account for a change in the β alone, the α remaining constant.

I give herewith the annuity values and the premiums for an ordinary life policy for an excess mortality of 25 per cent., 50 per cent., 75 per cent. and 100 per cent., and for comparison the same for the standard life. I also give for each case the increase in the age and the new rate of interest. The values are based upon Mr. Hunter's Makehamized American Table, 3 per cent. I also include in this table the results for a certain type of overweight lives referred to later in this paper.

I have plotted the six series of values of P_x here given. I have also included for comparison a case where the age alone is marked up. The first five curves evidently belong to the same system. Observe how impossible it would be to produce these results by

marking up the age alone as in curve VI. A change in the age alone would however describe fairly well the case of overweight.



I.

I now come to the treatment of the general case where the mortality cannot be adequately described by either an additive or a

multiplicative constant. And in treating this general case I shall appeal for an example directly to Volume II of the *Medico-Actuarial Mortality Investigation.*

I select the group on page 105, men of age 35–39 at entry, height 5 feet 11 inches to 6 feet 2 inches, and between 50 and 60 pounds overweight. I throw out the first two years on account of the supposed effect of medical selection and the last three years because the data are too few. This leaves the experience for 19 years on a life 38.8 years of age at the beginning of the period (36.8 being the average age at entry).

The problem is to find the α' and β' for this life, and, incidentally, to judge whether we are making a grave mistake by using the standard γ.

For this purpose I use the method of least squares. From the columns giving the deaths and the exposed to risk I can compute columns of l_x and d_x and hence μ_x by the formula

$$\mu_x = \frac{d_{x-1} + d_x}{2l_x}.$$

I give in Table II columns of the deaths, the exposed to risk and the μ_x.

This gives 18 values of μ_x beginning with the age 39.8. I have therefore to determine α' and β' so that

$$\sum_{x=39.8}^{x=56.8} (\mu'_x - \alpha' - \beta' e^{\gamma x})^2$$

shall be a minimum, where the values, μ'_x, are those given in the table above.

I will first, however, treat the general case in which the age is a and the number of years for which data are given is n. Then

$$\sum_{a}^{a+n-1} (\mu'_x - \alpha' - \beta' e^{\gamma x})^2$$

must be a minimum. For convenience I will drop the primes and the limits of summation.

Differentiating with regard to α and β and setting the results equal to zero, we have the following equations of condition:

$$\begin{aligned} n\alpha + \Sigma e^{\gamma x}\beta &= \Sigma \mu_x, \\ \Sigma e^{\gamma x}\alpha + \Sigma e^{2\gamma x}\beta &= \Sigma e^{\gamma x}\mu_x. \end{aligned} \tag{18}$$

Solving these equations for α and β we have:

$$\alpha = \frac{\Sigma\mu_x \Sigma e^{2\gamma x} - \Sigma e^{\gamma x}\Sigma e^{\gamma x}\mu_x}{n\Sigma e^{2\gamma x} - \Sigma^2 e^{\gamma x}}, \quad \beta = \frac{n\Sigma e^{\gamma x}\mu_x - \Sigma\mu_x \Sigma e^{\gamma x}}{n\Sigma e^{2\gamma x} - \Sigma^2 e^{\gamma x}}.$$

By performing the indicated summations we have:

$$\Sigma e^{\gamma x} = e^{\gamma a}\left(\frac{e^{n\gamma}-1}{e^{\gamma}-1}\right),$$

$$\Sigma^2 e^{\gamma x} = e^{2\gamma a}\left(\frac{e^{n\gamma}-1}{e^{\gamma}-1}\right)^2,$$

$$\Sigma e^{2\gamma x} = e^{2\gamma a}\left(\frac{e^{2n\gamma}-1}{e^{2\gamma}-1}\right).$$

Let

$$\Sigma\mu_x = F;$$

let

$$\Sigma e^{\gamma x}\mu_x = e^{\gamma a} G_a,$$

where

$$G_x = \mu_x + e^{\gamma}\mu_{x+1} + e^{2\gamma}\mu_{x+2} + \cdots + e^{(a+n-1-x)\gamma}\mu_{a+n-1}.$$

G_a (or G) can be computed by means of the recursion formula:

$$G_x = \mu_x + e^{\gamma} G_{x+1}. \tag{19}$$

Let

$$\frac{e^{n\gamma}+1}{e^{\gamma}+1} = A, \quad \frac{e^{n\gamma}-1}{e^{\gamma}-1} = B.$$

Then:

$$\alpha = \frac{AF - G}{nA - B}, \quad \beta = \frac{\dfrac{nG}{B} - F}{c^a(nA - B)}. \tag{20}$$

Since $e^{\gamma} = c$ we have the following expressions for A, B, F and G:

$$\begin{aligned} A &= \frac{c^n + 1}{c + 1}, \\ B &= \frac{c^n - 1}{c - 1}, \\ F &= \Sigma\mu, \\ G &= \mu_a + c\mu_{a+1} + \cdots + c^{n-1}\mu_{a+n-1}. \end{aligned} \tag{21}$$

A and B are independent of the μ's and of a, and, if the n remains unchanged, can be computed once for all and kept for repeated use.

F is got by addition of the μ's and G by a simple recursion method.

In the case in hand where $a = 39.8$ and $n = 18$, and the μ's have the values given above we obtain, by application of these formulas,

$$\alpha' = .00070344,$$
$$\beta' = .00009803.$$

The values of the standard α and β, namely for Mr. Hunter's Makehamized American Table, for which the constants are:

$$\log c = .04579609,$$
$$\log s = -.003296862,$$
$$\log g = -.00013205,$$

are

$$\alpha = .00759132,$$
$$\beta = .00003206.$$

We see, therefore, that the β of our sub-standard life is larger than standard and the α is smaller.

These values of α' and β' give an increase of age of 10.6 years but a decrease of the interest rate from 3 per cent. to 2.293 per cent. The values of a_x and P_x for this case are given in Table I.

In order to gain a better insight into this whole matter let us put it into graphic form. The expression $\mu_x = \alpha + \beta e^{\gamma x}$, when μ_x is plotted as ordinate and x as abscissa, is an ascending exponential curve in which the slope is continually increasing with x. Now suppose that we make a transformation of the coordinate plane by means of the equation $x' = e^{\gamma x}$, that is every point which had the abscissa x will now have the abscissa $e^{\gamma x}$, that is the plane is stretched x-wise in the ratio $e^{\gamma x} : 1$. If we identify the points x' with the points x it will be equivalent to plotting upon a coordinate surface in which the x scale is not uniform but increases exponentially.

I give herewith a diagram of this kind. The scale has been chosen in such a way as to exhibit the part of the plane between $x = 35$ and $x = 60$.

On a plane of this kind I have plotted three sets of values of μ_x. Those forming the line *I–I* are the μ's of Mr. Hunter's Makehamized American Table. Those forming the line *II–II* are the q's got from the M. A. standard reference table* on page 89 of Volume

* It was not thought worth while to compute the μ's as the difference for graphical purposes is not great.

II.

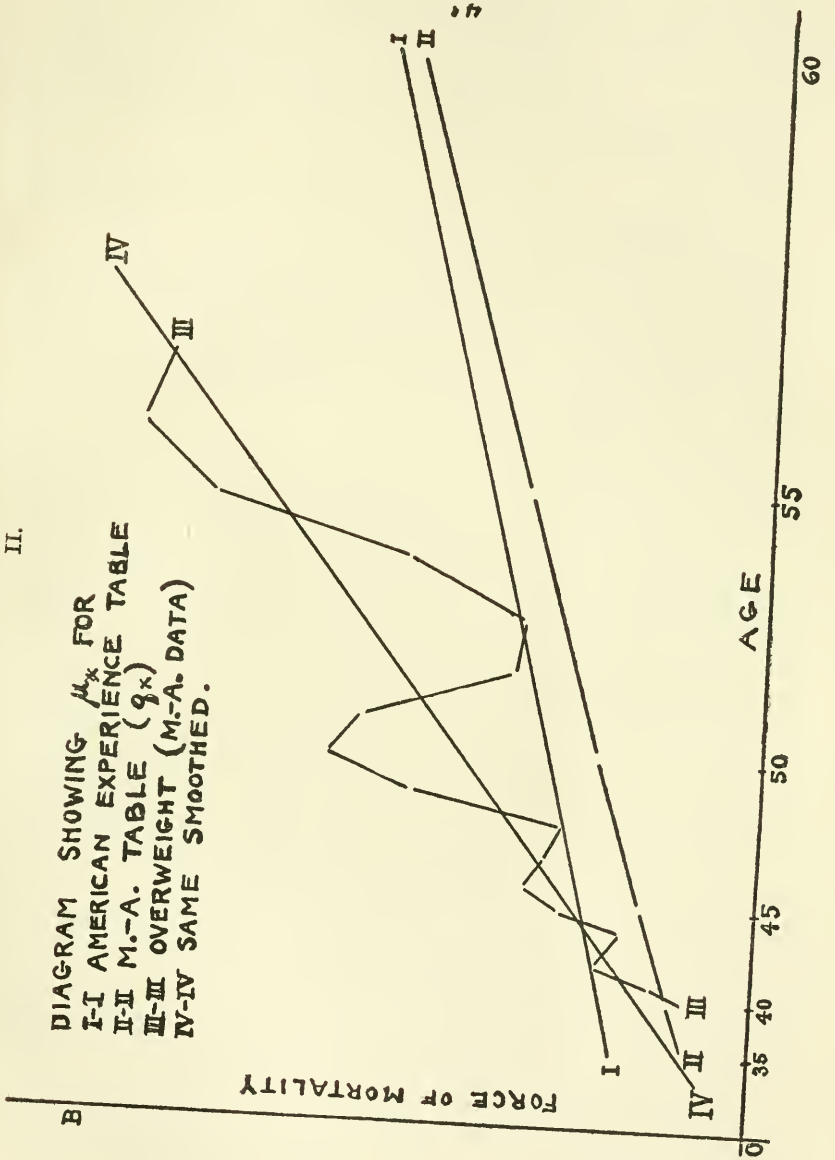


DIAGRAM SHOWING μ_x FOR
I-I AMERICAN EXPERIENCE TABLE
II-II M.-A. TABLE (9_x)
III-III OVERWEIGHT (M.-A. DATA)
IV-IV SAME SMOOTHED.
FORCE OF MORTALITY
AGE
0
35
40
45
50
55
60
I
II
III
IV

I of the *Medico-Actuarial Mortality Investigation.* The closeness with which these points follow a straight line would seem to give internal evidence that the M. A. Table was smoothed by the Makeham formula. It will be noticed that the fact is clearly exhibited here that the mortality of the M. A. Table is less than that of the American Experience Table for young ages but approaches it as the age increases.

The series of points forming the broken line *III–III* are the result of plotting the μ'_x's for the case here given in Table II, namely for 50–60 pounds overweight.

We have in this representation a simple graphical interpretation of the α' and β' of our life. α' and β' are namely the two parameters of the line. α' is ordinate of the point where the line crosses the axis OB and β' is the actual slope of the line measured in the chosen coordinate units.

The method of least squares has given us the α and β of the line that best fits the series of points *III–III.* The line *IV–IV* is this line plotted from the values $\alpha' = .00070344$, $\beta' = .00009803$.

Now several things are noticeable, first that the points *III–III* really do indicate fairly well a straight line; the deflections seem to be accidental variations due to paucity of data rather than to indicate any fundamental curvature. This is the justification which was hinted at for employing the standard γ.

If the real γ had been larger than the standard γ the points *III–III* instead of indicating a straight line would have indicated a curve concave upward and if the real γ had been less than the standard γ they would have indicated a curve concave downwards.

Even if they had indicated a curve, however, we should have been amply justified on the ground of practical considerations in using the standard γ; for this reason: the practical exigencies of the case do not warrant us in going beyond a linear treatment, since, in the first place, the actuarial work involved in a linear treatment is entirely practical, in the second place, to take into account variations in the γ immensely complicates the situation, thirdly, because the variations of γ are likely to be slight, and fourthly, because two parameters are ample even in the case of a non-standard life to give a fit that is perfect enough for the nature of the data.

As a practical matter I believe we even might dispense with the method of least squares and substitute the plotting method here given, although the application of the method of least squares is

not onerous. As the further results of the M. A. Investigation become available it is evident what should be done,—the α and β should be determined for each class of impairment in the hope of being able thus to lay the foundations for a parametric evaluation of impaired lives. Special exponentially ruled plotting paper should be prepared. The work might further be abridged by plotting the q's instead of the μ's without any serious sacrifice of accuracy.

I might say that I do not see any reasons why the insuring of sub-standard lives should not eventually take its place in life insurance practice on an equally good footing with the standard business. There do not seem to be any serious actuarial difficulties. To make the actuarial accommodations is a relatively trifling matter; commutation columns must be prepared for a number of rates of interest. In order to avoid troublesome interpolations these tables should be prepared for differences of one tenth of one per cent. for the range between 2 and 4 per cent. or thereabouts. It would be advisable also to have the tabular difference for age one tenth of a year instead of one year.

A more difficult problem is that of getting the data. Fortunately the results of the M. A. Investigation are about to be available. Beyond this it must be a matter of careful feeling ahead with a gradual acquisition of knowledge.

Life underwriters have not had occasion to exercise, to any great extent, what fire underwriters mean by underwriting judgment. By underwriting judgment I understand a judgment as to whether a risk is fairly typical of the particular class for which experience is available. Life underwriting judgments have been qualitative—"yes" or "no?" If the sub-standard business is to be done they must be more than this—they must be qualitative—"how much?" The judgments in fire insurance are far more difficult than those with regard to sub-standard life conditions, yet aside from the conflagration hazard the fire insurance business, when properly done is fairly stable in spite of many adverse conditions which do not affect the life underwriter. I believe the field of underwriting judgment can with safety be greatly extended and must be to meet modern sociological conditions.

In conclusion, I wish to add the observation that this method is likely to prove valuable for the purpose of group insurance and second, to call attention to the fact that while the Makeham con-

TABLE I.

A Table Giving a_x and P_x for Various Percentages of Increased Mortality, and for Overweight; Rate of Interest 3 Per Cent.

Age.	Percentage of Increased Mortality.										Overweight.	
	0 (Standard).		25%.		50%.		75%.		100%.			
	A_x	P_x	A_x	P_x	A_x	P_x	A_x	P_x	A_x	P_x	A_x	P_x
20	21.98	.01439	20.92	.01650	19.98	.01855	19.14	.02054	18.37	.02249	——	——
25	21.12	.01609	20.06	.01836	19.14	.02054	18.31	.02265	17.57	.02471	——	——
30	20.10	.01827	19.05	.02075	18.14	.02312	17.34	.02541	16.62	.02764	——	——
35	18.92	.02108	17.87	.02387	16.97	.02652	16.19	.02906	15.48	.03154	17.26	.02564
40	17.56	.02475	16.51	.02797	15.63	.03102	14.86	.03395	14.18	.03677	15.23	.03250
45	16.02	.02962	14.98	.03346	14.10	.03709	13.35	.04055	12.69	.04390	13.08	.04189
50	14.31	.03618	13.28	.04091	12.42	.04538	11.69	.04966	11.06	.05377	10.89	.05490
55	12.46	.04515	11.45	.05121	10.62	.05693	9.93	.06240	9.33	.06768	8.74	.07350
60	10.52	.05767	9.55	.06569	8.76	.07328	8.12	.08055	7.57	.08758	6.71	.10046
65	8.56	.07547	7.65	.08644	6.94	.09688	6.35	.10688	5.86	.11656	4.91	.14015
Increase of age.....			2.116		3.845		5.307		6.573		10.6	
New interest rate.....			3.196%		3.392%		3.588%		3.785%		2.293%	

TABLE II.

A Table of μ_x for a Certain Group of Overweight Lives.

Age.	Exposed to Risk.	Deaths.	μ_x.
38.8	2335	9	——
39.8	2097	10	.00434
40.8	1878	16	.00671
41.8	1662	20	.01044
42.8	1487	10	.00952
43.8	1336	14	.00872
44.8	1164	17	.01278
45.8	1002	15	.01512
46.8	808	10	.01396
47.8	693	10	.01368
48.8	572	17	.02286
49.8	483	12	.02844
50.8	394	10	.02609
51.8	299	2	.01647
52.8	247	6	.01590
53.8	190	4	.02347
54.8	148	7	.03611
55.8	100	3	.04105
56.8	45	2	.03944

stants in this, and, particularly, in the preceding paper, have been given a suggestive interpretation, the use made of them is essentially empirical and statistical and is in no wise dependent upon any particular theory as to their correlations with physical facts.

Graduation by the Summation Method. Some Elementary Notes.

BY

JOSEPH B. MACLEAN.

The method of graduation of mortality and other tables by summation has been made use of in several recent contributions to the *Transactions,* and it has occurred to me that as the method is one not very well known to students, a paper giving some account of it would be of use. The various papers dealing with the subject are scattered through many volumes of the *Journal of the Institute of Actuaries,* some of which are inaccessible to many of our students, and it may be thought an advantage to have some information on the subject available in the pages of our own *Transactions.* I also believe that, although no graduation by summation has been made in connection with any important mortality investigation of recent years (with the exception perhaps of the mortality tables based on the experience of The Manchester Unity of Oddfellows, 1893–97), the system is one of great utility and convenience, being especially adapted to the adjustment of sickness tables, and has, in some circumstances, a great deal to recommend it. I shall proceed in the first place to a brief statement of the principles underlying the use of formulas of summation and at a later stage take up some of the main questions connected with this subject. It should be stated here, however, that Mr. G. J. Lidstone has given a very full description of the basis of the summation method (*J. I. A.,* XLI, 348, and XLII, 106) to which all students should refer and the recent papers of Mr. King and Mr. Spencer (*J. I. A.,* XLI) also contain much valuable information. A good deal of what follows is taken from these papers, where however the subject is dealt with in a somewhat less elementary fashion. It is also convenient for completeness to repeat briefly the main points.

If the law of mortality were known, graduation would be unnecessary, because exact tables of any function according to that law could be prepared. The law of mortality is, however, not known, but mathematical expressions have been derived which are found to

give a close representation of the law experienced in the past among observed bodies of lives. We have every reason to suppose that mortality functions are subject to a definite law of progression and are therefore amenable to interpolation. For the purposes of graduation by summation, the assumption is made that successive groups of terms, representing the function dealt with, form series of the nth degree, so that $(n+1)$th and higher differences vanish, n generally being taken as three. As may be supposed, such an assumption has not passed without question, and, in fact, a very active controversy was at one time carried on as to whether the error admittedly introduced, since the true series did not, in fact, follow exactly such a law, did not counteract any advantages claimed for the system. It will be sufficient here to state that the error introduced by the assumption of constant differences over short sections of the table is of very little practical importance and may be neglected.

The rough series we have to graduate may, therefore, be considered as consisting of two parts, one of which is a regular series proceeding by constant third differences (say), and the other an irregular series consisting of the errors of observation, some of which will be positive and some negative. The summation method of graduation aims at applying an arithmetical process which shall reproduce the former series and effect a redistribution of the terms of the latter. This arithmetical process is, in general, to substitute for an ungraduated value of the second series a value made up of the ungraduated value and the surrounding values taken in various proportions. Thus the method does not profess to eliminate error, but only to spread it more evenly over all the terms. As the terms of the series of errors are not all of the same sign, it is clear that this process will however, in general, tend to effect a reduction in total amount. If the true series be assumed to proceed by constant third differences, then the arithmetical process must be such as to reproduce a third difference series* and the problem is to find a method of selecting the different proportions of the terms to be included in the graduated value.

A short account of the different summation processes which have been employed is now given as a consideration of these is the best

* Sometimes it is found advisable, however, in order to secure greater smoothness, to introduce a slight theoretical error in addition to that involved in the assumption of third differences as, for example, in Hardy's "Friendly Society" formula.

way to understand the practical application and to appreciate the theoretical points involved.

Finlaison's Method.

In 1829 a mortality table based on the experience of Government life annuitants and other data was prepared for the use of the British Government by John Finlaison, and a description of the construction of that table is contained in a Government report published in that year. This report gives a very short description of the first graduation of the summation type.

The function graduated was the probability of living a year at each age, and the process employed was equivalent to summing the unadjusted values of p_x in groups of five twice, placing the result in each case opposite the central age of the group, the adjusted values then being obtained by dividing the numbers in the second summation by 25, that being the number of unadjusted terms included in each. In Mr. G. F. Hardy's notation these operations are denoted by

$$p'_x = \frac{[5]^2}{25} p_x,$$

where p_x is an unadjusted and p'_x an adjusted value. It will be seen that the result of the first summation is to place opposite age x the value of

$$p_{x-2} + p_{x-1} + p_x + p_{x+1} + p_{x+2},$$

and after the second summation

$$p_{x-4} + 2p_{x-3} + 3p_{x-2} + 4p_{x-1} + 5p_x + 4p_{x+1} + 3p_{x+2} + 2p_{x+3} + p_{x+4}.$$

Finlaison's only comment on this formula is that it is "perhaps quite as good, but more laborious," than another method which he used for the graduation of some other tables. The other method referred to is to obtain the graduated value of p_x from the formula

$$\begin{aligned} p'_x = \sqrt[3]{}\{&(p_{x-3} \times p_{x-2} \times p_{x-1} \times p_x \times p_{x+1})^{\frac{1}{5}} \\ &\times (p_{x-2} \times p_{x-1} \times p_x \times p_{x+1} \times p_{x+2})^{\frac{1}{5}} \\ &\times (p_{x-1} \times p_x \times p_{x+1} \times p_{x+2} \times p_{x+3})^{\frac{1}{5}}\}. \end{aligned}$$

This is the same thing as summing the logarithms of p_x in groups

of five and the resulting series in groups of three, dividing then by 15, or in symbolic form,

$$\log p'_x = \frac{[5][3]}{15} \log p_x,$$

where p' is the adjusted value. This latter process does not seem to effect any appreciable saving in labor, and it would appear that Finlaison did not carry out his graduation by summing as above described, but used some other less direct method. The latter process would, of course, result in a different graduation, as it involves only seven instead of nine terms of the original series in each summation.

If we expand the expression for the graduated value of p_x by the first method, viz.,

$$p_{x-4} + 2p_{x-3} + 3p_{x-2} + \text{etc.},$$

in terms of p_x and its differences by the formula,

$$p_{x+n} = p_x + n\Delta p_x + \frac{n(n-1)}{2} \Delta^2 p_x + \text{etc.},$$

we shall obtain,

$$\frac{[5]^2}{25} p_x = p_x + 2\Delta^2 p_x + \text{etc.}$$

This process, therefore, will reproduce only a series proceeding by constant first differences, and if it be applied to a third difference series there will be an error introduced, in addition to errors of observation, of approximately twice the second difference of each term in the true series, subsequent differences being small. The method therefore will give only a very rough representation of the truth as regards this series and will, in fact, distort it somewhat, while the effect on the series of error is evidently to distribute each term among the surrounding terms in the proportion indicated by the successive coefficients.

Although this formula is far from perfect, it is very simple in application and has been used on several occasions where a more or less rough graduation was sufficient. It has also been employed where the nature of the data did not justify a more refined process, and notably by Mr. King in the adjustment of his "Analyzed Tables" (*J. I. A.*, XIX, 387, see also *J. I. A.*, XVIII, 168). In general the formula tends to produce a flat mortality curve, as

would be expected from the considerations above. Subsequent improvements have been in the direction of reducing the theoretical error and in providing a formula having a series of numerical coefficients which effect a more suitable distribution of the errors of observation as explained hereafter.

Woolhouse's Formula.

In 1869 Woolhouse graduated the Institute of Actuaries' Life Tables by a method which was correct to third differences and which, therefore, was a great advance over Finlaison's method in point of accuracy of reproduction of the underlying mortality curve. His process, briefly stated, was to take for the adjusted value of d_x the arithmetical mean of five series, each of which was obtained by third difference interpolation of a different set of quinquennial values of the original series. He did not, at that time, know that his results could be obtained by a process of summation, his formula being in fact only a particular case of a general type, and his practical procedure was needlessly laborious.

As Woolhouse's formula is of some importance in connection with the subject of graduation, and as a description of it can only be obtained from an early volume of the *Journal of the Institute of Actuaries* or in the *Institute of Actuaries Life Tables,* it may be of advantage to give here a short demonstration of the formula in its original shape for the benefit of those students who are unable to refer to these volumes.

As stated above, the process was to obtain, in effect, five different series of interpolated values based on the five following series of unadjusted terms,

$$\begin{array}{cccc} d_{10}, & d_{15}, & d_{20}, & \cdots \\ d_{11}, & d_{16}, & d_{21}, & \cdots \\ \vdots & \vdots & \vdots & \\ d_{14}, & d_{19}, & d_{24}, & \cdots \end{array}$$

the final graduated value of any term being the arithmetical mean of the five values so obtained. The formula of interpolation was

$$d_{x+t} = d_x + \frac{t}{5}\, a_x + \frac{1}{2}\left(\frac{t}{5}\right)^2 b_x,$$

the interval of differencing being five, and a_x and b_x being the first

and second central differences of d_x. It is easily seen that the value of a_x in terms of d_x is $\frac{1}{2}(d_{x+5} - d_{x-5})$, and of b_x, $(d_{x+5} - 2d_x + d_{x-5})$, so that

$$d'_{x+t} = d_x + \frac{t}{10}(d_{x+5} - d_{x-5}) + \frac{t^2}{50}(d_{x+5} - 2d_x + d_{x-5})$$

$$= -\frac{t(5-t)}{50} d_{x-5} + \frac{25 - t^2}{25} d_x + \frac{t(5+t)}{50} d_{x+5}.$$

It is not, however, necessary to make five separate interpolations in each case, as we may at once obtain from the last formula an expression for the final graduated value of d_x by giving t successively the values 2, 1, 0, -1, and -2, while x is written $(x-2)$, $(x-1)$, x, $(x+1)$, and $(x+2)$. The average of the five values of d_x so obtained is evidently the result required. Proceeding in this way we have successively, where d'_x is an interpolated and d_x an ungraduated value,

$$d'_x = -.12d_{x-7} + .84d_{x-2} + .28d_{x+3},$$
$$d'_x = -.08d_{x-6} + .96d_{x-1} + .12d_{x+4},$$
$$d'_x = \qquad\qquad d_x,$$
$$d'_x = +.12d_{x-4} + .96d_{x+1} - .08d_{x+6},$$
$$d'_x = +.28d_{x-3} + .84d_{x+2} - .12d_{x+7}.$$

Summing, we have

$$5d'_x = d_x + .96(d_{x-1} + d_{x+1}) + .84(d_{x-2} + d_{x+2}) + .28(d_{x-3} + d_{x+3}) + .12(d_{x-4} + d_{x+4}) - .08(d_{x-6} + d_{x+6}) - .12(d_{x-7} + d_{x+7}),$$

i. e.,

$$d'_x = .2d_x + .192\gamma_1 + .168\gamma_2 + .056\gamma_3 + .025\gamma_4 - .016\gamma_6 - .025\gamma_7,$$

where $\gamma_n = (d_{x-n} + d_{x+n})$.

This is also useful as showing the general form of all summation formulas when expanded in terms of the ungraduated values, when it is seen that the terms equidistant from the central term are symmetrically involved. This is a perfectly general relation and we may write for all summation formulas, where u'_x is a graduated and

u_x an ungraduated value,

$$u'_x = au_x + b\gamma_1 + c\gamma_2 + \text{etc.},$$

where $\gamma_n = (u_{x-n} + u_{x+n})$.

Mr. G. F. Hardy and Mr. Ackland have given methods of obtaining Woolhouse's results by summation (*J. I. A.*, XXIII and XXXII) to which it may be interesting to refer. The process described by Mr. Hardy (*J. I. A.*, XXIII, 251) depends on the fact that it is the same thing whether we sum a series three times in fives or difference the same series three times negatively in intervals of five and then sum the resulting series continuously from the bottom upwards, repeating the process twice. The results produced are the same, as may be seen from the following. Let the series be u_0, u_1, u_2, etc., so that the result of differencing negatively in intervals of five is to produce the series

$$\begin{gathered} u_0 - u_5, \\ u_1 - u_6, \\ u_2 - u_7, \\ \text{etc.} \end{gathered}$$

Differencing again in the same way we have,

$$\begin{gathered} u_0 - 2u_5 + u_{10}, \\ u_1 - 2u_6 + u_{11}, \\ u_2 - 2u_7 + u_{12}, \\ \text{etc.}, \end{gathered}$$

and finally after the third process,

$$\begin{gathered} u_0 - 3u_5 + 3u_{10} - u_{15}, \\ u_1 - 3u_6 + 3u_{11} - u_{16}, \\ u_2 - 3u_7 + 3u_{12} - u_{17}, \\ \text{etc.} \end{gathered}$$

The sum of all the terms of this last series is evidently

$$(u_0 + u_1 + \cdots + u_4) - 2(u_5 + u_6 + \cdots + u_9) + (u_{10} + u_{11} + \cdots + u_{14}),$$

other terms cancelling, and this would be the top term in a continuous summation from the bottom upwards, the form being the same for other terms of the new series. A second continuous sum-

mation will produce terms of the form

$$u_0 + 2u_1 + 3u_2 + 4u_3 + 5u_4 + 3u_5 + u_6 - u_7 \\ - 3u_8 - 5u_9 - 4u_{10} - 3u_{11} - 2u_{12} - u_{13},$$

and a third,

$$u_0 + 3u_1 + 6u_2 + 10u_3 + 15u_4 + 18u_5^* + 19u_6 + 18u_7 \\ + 15u_8 + 10u_9 + 6u_{10} + 3u_{11} + u_{12},$$

which is exactly the form of the terms obtained by summing the original series thrice in fives.

Now it may be shown by other methods (referred to hereafter) that Woolhouse's method was equivalent to performing the operations

$$\frac{[5]^3}{125}(u_0 - 3b_0)$$

(*J. I. A.*, XXXII, 372). Hence in Mr. Hardy's method there remains to be deducted three times the central second difference (or three times the second ordinary difference of the preceding term). As a matter of fact, Mr. Hardy, starting with the ungraduated l column, obtains the graduated value of d_x and for this reason makes only two continuous summations instead of three.

Mr. Ackland's process is simpler. He compared Mr. Woolhouse's expression for the graduated value with that obtained by three summations in fives and then expressed the difference between them in summation form. It is unnecessary to repeat the details.

Mr. Todhunter has shown (*J. I. A.*, XXXII, 389), that the general expression for the graduated value, when the arithmetical mean of n third difference interpolations based on original quinquennial values is taken, is,

$$\left[\frac{3n^2+5}{8n^3}[n]^3 - \frac{n^2-1}{8n^3}[n]^3[3]\right]u_x,$$

where $[n]$ denotes the sum of n terms and $[n]^m$ denotes the operation of summing in groups of n terms m times.

When $n=5$ this becomes

$$\frac{1}{125}\{10[5]^3 - 3[5]^3[3]\}u_x.$$

Formulas of J. A. Higham.

It was with a view to combine the supposed accuracy of Woolhouse's method with the simplicity in application of Finlaison's that the formulas of J. A. Higham were constructed, and the series of papers commencing in the twenty-third volume of the *Journal of the Institute of Actuaries* marks the first advance in the theory of this subject. Higham's method of obtaining summation formulas correct to third differences may first be illustrated simply as follows.

Selecting an odd number of terms of the series to be graduated, such as nine or fifteen, we may find two different sets of summations which will each reduce the terms selected to one term. For example, if nine original terms be taken, then a single summation of the nine terms or two summations in fives will each result in a single term. If the terms selected be denoted by u_0, u_1, $\cdots$, u_8, then the single term produced in the first case is

$$u_0 + u_1 + u_2 + \cdots, + u_8,$$

and in the second case it is

$$u_0 + 2u_1 + 3u_2 + 4u_3 + 5u_4 + 4u_5 + \cdots, + u_8.$$

Denote these two expressions by $[9]u_4$ and $[5]^2u_4$ respectively. By expanding the central term and each of those summations in terms of u_0 and its differences so far as Δ^3 (if ordinary differences be employed), we shall obtain three equations involving the unknown quantities Δu_0, $\Delta^2 u_0$, and $\Delta^3 u_0$, from which, by elimination, an equation expressing the value of the central term in the form

$$u_4 = m[9]u_4 + n[5]^2u_4 \qquad (A)$$

is obtained, m and n being numerical coefficients which may be positive or negative. This result means that, if we take a series proceeding by constant third differences and sum in nines, and also separately twice in fives, then m times the term in the first resulting series opposite any age, added to n times the corresponding term in the second resulting series, will reproduce the original value. If an irregular series not proceeding by third differences has been superimposed on the regular series, it will not be reproduced by this process, but its terms will be redistributed in proportion to the respective coefficients of u_0, u_1, etc., in formula (A) expanded. In general, the process consists in combining multiples of different

summations of the same set of terms (which summations have first been expressed in terms of the first term and its differences) in such a way as to eliminate the differences. A numerical example may help the student to understand this process and we may take the following from one of Higham's papers. Selecting fifteen original terms we find that by summing them in groups of three and then in groups of five three times, or, alternatively, by summing in fives twice and then in sevens, we shall reduce the fifteen terms to a single term.

Now

$$\begin{aligned}[3][5]^3u_7 &= 55u_7 + 52(u_6 + u_8) + 43(u_5 + u_9) \\ &\quad + 31(u_4 + u_{10}) + 19(u_3 + u_{11}) + 10(u_2 + u_{12}) \\ &\quad + 4(u_1 + u_{13}) + (u_0 + u_{14}) \\ &= 375u_0 + 2625\Delta u_0 + 9125\Delta^2 u_0 + 20625\Delta^3 u_0,\end{aligned}$$

or

$$\frac{[3][5]^3}{375}u_7 = u_0 + 7\Delta u_0 + 24\tfrac{1}{3}\Delta^2 u_0 + 55\Delta^3 u_0.$$

Similarly

$$\frac{[5]^2[7]}{175}u_7 = u_0 + 7\Delta u_0 + 25\Delta^2 u_0 + 59\Delta^3 u_0,$$

also

$$u_7 = u_0 + 7\Delta u_0 + 21\Delta^2 u_0 + 35\Delta^3 u_0.$$

From these three equations, by elimination of Δu_0, $\Delta^2 u_0$, and $\Delta^3 u_0$, we may obtain the formula

$$u_7 = \frac{16[3][5]^3}{1000} - \frac{2[5]^2[7]}{\quad}u_7,$$

which will reproduce a series of the third degree.

It is obvious that we could, in a similar manner, obtain formulas which would reproduce series of terms of a higher degree than the third, but the formulas are generally unwieldy and it is doubtful if any real advantage is obtained.

The selection of the number of terms to be summed (fifteen in the example given above) is arbitrary, and the summations which can be made of a given set of terms, in the manner described, are sometimes numerous so that a large number of formulas similar to

formula (A) could be obtained, any one of which would reproduce the underlying series and effect a redistribution of the series of error. The choice is however limited by practical conditions: the coefficients m and n must be such as to permit facility of calculation; a sufficient, but not excessive, number of terms of the original series to secure effective graduation must be embraced; and the coefficients of the successive terms in the expansion of the resulting formula must be such as to give due, but not undue, prominence to the central terms.

Methods of G. F. Hardy.

In 1896 Mr. G. F. Hardy published a well-known paper on this subject (*J. I. A.*, XXXII, 371) in which he gave an alternative method of obtaining graduation formulas similar to those of Higham. Mr. Hardy worked out his formulas in terms of the central value instead of the initial value used by Mr. Higham and the various summations do not always include the same number of terms. His mode of procedure will be seen from the following. If $[n]u_0$ denote the sum of n terms of a series of which u_0 is the central term and of which, therefore, $u_{-\frac{n-1}{2}}$ and $u_{+\frac{n-1}{2}}$ are respectively the first and last, n being odd, then by an elementary formula of central differences, we shall have, neglecting fourth and higher differences,

$$[n]u_0 = n\left\{u_0 + \frac{n^2-1}{24}b_0\right\} = nu_0\left\{1 + \frac{n^2-1}{24}b\right\}, \qquad B)$$

where b is a symbol of operation.

Thus

$$[3]u_0 \text{ (or } u_{-1} + u_0 + u_{+1}) = 3u_0\{1 + \tfrac{1}{3}b\}.$$

With the aid of the above formula (B) we may write, generally,

$$[n][m][p]\cdots u_0 = nmp\cdots u_0\left\{1 + \frac{n^2 + m^2 + \cdots - t}{24}b\right\}.$$

Since

$$[n][m][p]\cdots u_0 = nm\cdots u_0\left\{1+\frac{n^2-1}{24}b\right\}\left\{1+\frac{m^2-1}{24}b\right\}\cdots \text{etc.}$$

$$= nm\cdots u_0\left\{1 + \frac{n^2 + m^2 + \cdots - t}{24}b\right\},$$

terms involving higher powers of b vanishing; where $[n][m][p]\cdots u_0$ denotes a summation in groups of n followed by one in groups of m and so on, and t is the number of the summations. The numerical coefficient of b will, in every set of summations of the type shown above, be positive, and if we can find a different way of grouping the summations which will result in a negative coefficient of b, the two sets may then be combined to eliminate b provided that differences of a higher order than the third are assumed to vanish. Such a way of grouping summations is obviously that denoted by

$$[n][m]\cdots u_0 \quad \pm [p][q]\cdots u_0 \quad \pm \cdots, \text{ etc.},$$

since we can so choose n, m, p, q, etc., that the resulting coefficient of b when the expression is expanded by the formula given above, is negative and numerically equal to the coefficient of b obtained from the previous set of summations. For example, we have

$$([1] + [3] + [5] - [7])u_0 = 2u_0(1 - 4b),$$

while

$$[5]^2[7]u_0 = 175u_0(1 + 4b).$$

Hence, neglecting higher differences than the third, we shall have,

$$\frac{[5]^2[7]}{350}\{[1] + [3] + [5] - [7]\}u_0 = u_0,$$

which is Mr. Spencer's well-known twenty-one term formula.

In the foregoing, the expression within the brackets, namely, $\{[1] + [3] + [5] - [7]\}$, is sometimes referred to as the operand, while $[5]^2[7]$ is called the operator. In actual working it is usually simpler to obtain first the operand by a few simple steps which are obvious from its expanded form. Thus $\{[1] + [3] + [5] - [7]\}u_0 = 2u_0 + (u_{-1} + u_{+1}) - (u_{-3} + u_{+3})$. The summations contained in the operator are then applied.

Having written down a set of summations resulting in a negative coefficient of b, i. e., an operand, the various sets of summations, or the various operators, which will produce the same positive coefficient, can be determined by inspection of a table giving the values of the function $(n^2 - 1)$ or of $(n^3 - n)$ for practicable values of n. Several examples of the process are given in Mr. Hardy's paper.

The only recent development of the summation method to which

I shall refer is the process demonstrated by Mr. King (*J. I. A.*, XLI, 530) and also, in a different way, by Dr. Karup (*Transactions, Second International Congress of Actuaries*). Mr. King points out that Woolhouse's formula was deduced by interpolation, its summation form being subsequently discovered, and shows that it is possible similarly to obtain other formulas of summation by first obtaining the expanded expression for the graduated value in terms of the surrounding ungraduated values by various processes of interpolation. When this expanded expression has been obtained, he proceeds by general reasoning to show what are the possible summations involved, and, knowing the number of terms included, it is an easy matter to throw the expression into the summation form. By using osculatory interpolation and going as far as fifth differences, Mr. King arrives at formulas which would give exceedingly smooth results and which have a very small theoretical error, but not all of his formulas are suitable for practical application. Mr. Lidstone has since given a very simple and direct method of determining the operand having found the governing summations, by which the summation forms of different interpolation formulas are readily obtained in the simplest practical form (*J. I. A.*, XLII, 133).

Having now given some account of the development of the summation method and having examined the various ways of producing graduation formulas, we may turn to a consideration of some of the questions connected with the subject with which students may have difficulty. In the first place, when should a summation graduation be employed?

When the characteristics of a Makeham graduation are not desired, or when it is found that the data do not conform to Makeham's Law, the object of graduation is to produce a smoothly progressing series which shall reflect the feature of the ungraduated data, eliminating however, as far as possible, all errors. We have seen that, in general, this is the work accomplished by a summation formula. It does not follow, however, that it will do so in all circumstances. It may be said that the method is applicable only where the data are of sufficient extent throughout the main part of the table to ensure that the principle portion of the mortality curve will not be clouded over by accidental sequences of errors such as are likely to occur where the facts are few. As regards the ends of the table, it will always be the case that the facts are inadequate

upon which to base graduated results without special treatment, but this is generally true no matter what method of graduation be adopted. It is, however, the case that in the summation method the facts at extreme ages have an effect on the graduated values at other ages which is more or less important according as the number of terms summed is large or small, and special methods of dealing with these end terms are necessary. These methods can, however, be chosen with discretion, and, by the exercise of judgment, at least as good a table throughout the entire range of life can be got as results from the use of other methods. A notable example is Mr. Spencer's graduation of the British Government Female Annuity Experience, 1883 (*J. I. A.*, XLI, 361), where a summation formula produced much smoother results at extreme ages than were produced by Dr. Sprague using the graphic method, although the original facts at these ages were meagre and irregular.

Again, when the law of mortality is subject at some point to a rapid change, as in the years following selection, the summation process is not suitable and would not reproduce the true characteristics of the table in the part affected. Mr. King's graduation of his "Analyzed Tables" (*J. I. A.*, XIX, 387) has been criticized for this reason. The same thing would be true for the early years of life in the case of a population table, but in any case these ages would have to be specially dealt with. It would seem, however, that, where no such special feature is found, where the data are of sufficient quantity, and where Makeham's Law does not apply, the summation method has very great claims for adoption, and this has been recognized by actuaries whose opinion carries great weight. Thus Mr. G. F. Hardy says (*J. I. A.*, XXVII, 276): "Without attempting to determine generally between the merits of the two systems (graphic and summation), it may be said, that while the graphic method is, in some cases perhaps, the only method available, on the other hand, for the adjustment of ordinary mortality and sickness tables, the use of formulas, such as Mr. Woolhouse's, Mr. Higham's or that given below, is not open to any practical objection, and has an undoubted advantage in respect of the ease and rapidity with which they can be applied, and the certainty of the results obtained." Again, Mr. John Spencer says (*J. I. A.*, XLI, 376) ". . . I may be permitted to express the view that the utility of mechanical methods of graduation has not yet been fully recognized, and that their possibilities are far from being exhausted. It is

true that graduation by summation is not always feasible, and in some cases, even when the method can be satisfactorily applied, it may, in order to subserve practical ends, be preferable to rely upon some alternative method of graduation based on the assumption that the series to be graduated follows some mathematical law. At the same time, if the facts are sufficiently well distributed to admit of the application of any but a purely analytical method of graduation, a suitable summation formula will, with rapidity and precision, bring to light the smooth curve representing the law of the facts, with but a tithe of the labor attending the application of the graphic method, and with a fidelity throughout the table which is not to be looked for when a mathematical law of mortality is assumed to prevail."

It may be useful as bearing upon the question of when a summation method of graduation should be adopted to set forth in a concise form the principal advantages and disadvantages of the system. The main advantages are:

1. The production of a series which can be relied on to represent the facts and at the same time have the required degree of smoothness.

2. Facility of practical application.

3. Suitability when the graduated series has to be subdivided into two or more graduated series.

Regarding the first of these, the degree of smoothness attained depends on the particular formula employed, and, by the use of powerful formulas, results may be obtained which are as smooth as could be produced by the application of any other process. The fidelity with which tables graduated by formulas of summation reproduce the characteristics of the original facts, as measured by a comparison of the expected and actual deaths, is remarkable and it is interesting to notice that, to a certain extent, the various formulas are, in this respect, about on a par, notwithstanding that some are so much more accurate or more powerful in other respects than others.

Facility of application, while not of the first importance, is nevertheless a factor worthy of consideration and in this respect the summation method has especial advantages, since practically the entire work can be done by unskilled labor.

The third of the advantages mentioned above applies in the case of sickness tables where sickness applicable to different durations has to be separately tabulated as well as the whole amount. The

most extensive sickness tables at present in existence (Manchester Unity of Oddfellows Experience, 1893–97) were graduated in this way.

Finally it may be mentioned that if the function graduated is of the form $a + bx + cx^2 +$ etc., the summation method is correct in principle.

On the other hand, the method has certain disadvantages, chief of which are the following:

1. Loss of the property of "uniform seniority."
2. The tendency to convert an accidental sequence of error into a feature of the table.
3. Failure to give weight to the extent of the actual observations at each age.
4. Inapplicability of the method to the end terms.
5. Introduction of theoretical error.

The loss of uniform seniority would, in some cases, be fatal and in such circumstances the method is, therefore, inapplicable. Thus, for tables to be used in life insurance calculations and in which Makeham's Law had been found to apply, the summation method would not be considered.

The tendency mentioned in the second heading above is the greatest disadvantage of the system and is inherent. It is possible, however, by the use of powerful formulas to overcome it in a large degree, wherever it is considered necessary to do so.

Regarding the failure to give weight to the extent of the data at individual ages, it is usually the case that this is of importance only at very young and very old ages, and these ages require special treatment in any case. This requirement of special treatment at extreme ages is, perhaps, the second greatest disadvantage of the summation method. It is separately considered later. The disadvantage of the introduction of theoretical error has been referred to already and, as stated, is of very little importance.

Leaving now the question of when a summation graduation should be employed, let us consider what are the qualifications desirable in a graduation formula. The first qualification of a summation formula is that it shall have sufficient graduating power for the purpose in hand, that is to say, the power to produce a smooth series from the available data and one from which errors have been eliminated. Evidently the power to deal effectively with the errors depends on the extent and distribution of the errors in the un-

adjusted figures, and these may be found spread irregularly and with frequent change of sign throughout the table or they may come in groups of the same sign which, if not removed by a suitable formula, may be converted by graduation into a feature of the table.

Formulas most free from theoretical error are not necessarily the best for removing grouped irregularities, the number of terms included being of more importance. Short formulas, which are usually more accurate, involve the central values in greater proportion than long formulas and short formulas therefore allow each ungraduated value to have relatively more effect on the corresponding graduated value, and consequently tend to preserve to some extent any well marked feature of the unadjusted figures, accidental or otherwise. A formula, on the other hand, whose coefficients are represented by a flatter curve, i. e., one of which the central coefficient is not considerably larger than the coefficients of the surrounding terms, spreads each error over a larger number of graduated terms, thus giving less prominence to an accidental error of magnitude than the other class of formulas. In general, the coefficients of long formulas have a flatter curve over the central portion, and long formulas are therefore more powerful in dealing with grouped errors.

An important qualification for a graduation formula is that the coefficients of the expanded form should themselves form a smooth series. Assuming that the true underlying series remains unaltered by the process of graduation, it is easy to see that each term of error is gradually brought into the adjusted series by parts which are proportionate to the successive coefficients of the graduation formula. If then these coefficients themselves progress regularly, the graduated series of error will itself be a smoothly progressing series.* Various means of testing the smoothing power of graduation formulas have been suggested which may be relied on to give a good indication of the relative smoothness to be expected, but apart from the test of actual trial, none is entirely conclusive. Mr. G. F. Hardy suggested the comparison of the probable errors in the graduated and ungraduated figures, and gave examples based on the assumption that each ungraduated value was affected by a similar probable error (*J. I. A.*, XXXII, 376). The probable error in the ungraduated value is the amount by which it is as likely to exceed the true value as it is to fall short of it and is susceptible of

* For a more complete demonstration of this, see *J. I. A.*, XLI, 354.

calculation. The probable error in the graduated value may then be obtained from the expression for that value in terms of the ungraduated values, since the probable error in the sum of a number of terms is the square root of the sum of the squares of the individual probable errors. The assumption of equal probable errors is incorrect, as the terms at the extremes of all the series with which actuaries have to deal are usually subject to very much larger probable errors, on account of scarcity of data, than the intermediate terms. Mr. Lidstone, however, has demonstrated (*J. I. A.*, XLII, 124) that, on account of the symmetrical form assumed by the curve of these probable errors, the results arrived at by Mr. Hardy are valid. The ratio of the probable error in the third difference of the graduated value to that of the ungraduated value (calculated as stated above) is sometimes used as a measure of smoothing power, and this ratio has been termed by Mr. Spencer the "smoothing coefficient." As he points out, however, these criteria of smoothing power are not absolutely conclusive, as the sequences of errors may be such that a formula having a comparatively large smoothing coefficient (and which might therefore be expected *prima facie* to produce a less smooth graduation) may give, in a particular case, a better graduation than a formula having a smaller smoothing coefficient. In most cases, however, these coefficients may be relied on to give a good indication of the relative smoothness to be expected to result from the use of different formulas. Some illustrations may be found in one of Mr. Spencer's papers already referred to.

Formerly much attention was paid to the amount of theoretical error introduced into the smooth series, but it is now well recognized that this is of very minor importance as compared with the degree of smoothing power. The well known discussion between Dr. T. B. Sprague and Mr. Woolhouse, and the researches of Mr. King have established the fact that, except at extreme ages, the amount of such error is quite unimportant. Nevertheless, a good deal of attention has been paid to the question of obtaining formulas free from theoretical error. One of the most interesting of the formulas obtained is that given by Mr. Spencer for the graduation of colog p or μ. This was built up from a consideration of the form of the differential coefficients of these functions, assuming Makeham's Law, and by the insertion of a second difference error counteracting the errors arising from subsequent differences (*J. I. A.*, XLI, 402).

The necessary qualification of facile application has already been referred to. Many formulas could be produced which would satisfy all the theoretical requirements as to smoothing power and freedom from error, but which could never be made use of on account of arithmetical difficulties. Many other points arise for consideration in the choice of suitable graduation formulas, but a fuller discussion cannot be attempted here, and the student is referred for such a discussion to the valuable papers of Messrs. Spencer and Lidstone in the *Journal of The Institute of Actuaries.* It will be sufficient to say, in closing this section of the paper, that the choice is by no means an arbitrary one, but, on the contrary, offers scope for considerable skill and the exercise of sound judgment.

Having decided on the use of the summation method and having chosen a suitable formula, it still remains to decide what function shall be operated on. In the early days of the summation method the discussion of this question gave rise to rather heated argument, although the investigations of later writers seem to indicate that it is one of lesser importance, and it is certainly of very much less importance than the question of what formula shall be used. Higham graduated the rate of mortality, q_x, as did Sprague in applying the graphic method, but Woolhouse used the numbers living (or dying), and insisted that his method was specially designed for the graduation of that function and could not properly be applied to the adjustment of q_x, which, he said, was "not sufficiently amenable to interpolation over the intervals chosen." He also maintained that the numbers living were especially suitable for his method of adjustment, because, from the limiting age on, l_x may be taken as zero, and also because, in the successive interpolations, the same aggregate mortality as in the original facts is retained between any two points of contact. Mr. King, however, has observed that, having assumed constant third differences in the function dealt with, the values cannot accurately be taken as zero from the limiting age, while, as regards the retention of the same aggregate mortality as in the unadjusted data between two quinquennial points, it is at least doubtful if this is necessarily an advantage.

Recent investigations have shown that, as regards the amount of theoretical error introduced, certain functions are very much better than others, although the difference is only of moment at extreme old age. It is interesting, in view of Woolhouse's contention referred to above, to notice that, from this point of view, l_x is one of

the least suitable functions to graduate, while q_x is one of the most suitable for the summation method. The results of these investigations, however, lead one to the practical conclusion that, from the point of view of the amount of distortion caused by the assumption of constant differences, one function is about as good as another, since the error is, in any case, too small to be of any importance. Preferably the function graduated should be one of few figures, for arithmetical reasons. Reference may be made to the suggestion of Mr. Todhunter (which, by the way, does not seem ever to have been acted upon), that the function e_x should be adopted for graduation purposes, since this function in the unadjusted state already has a considerable degree of smoothness. Another suggestion is that the exposed to risk and actual deaths should be separately adjusted, and this would have the effect of giving some weight to the extent of the data at the various ages, but, on the other hand, we should lose the close degree of relationship which exists between these functions in the unadjusted state.

The last part of the subject which I shall take up is the method of dealing with the end terms. One of the real disadvantages of the summation method is that the application of a single formula necessarily leaves a greater or lesser number of terms at each end of the table ungraduated: if the number of terms included in the formula is $(2n+1)$, n terms at each end remain to be specially dealt with. Of course, the data at the extremes of the table are practically always of so little volume as compared with the main body of the table that any method of graduation generally requires special adjustment at the ends of the table. The graduated results, however they may be obtained, can never carry any great weight, but in a summation process involving a large number of terms the number of values requiring special treatment is correspondingly large, and these may extend to ages which would come well within the scope of the main graduation when other methods are adopted. At the young ages it will sometimes be the case, especially in insurance experiences, that the data extend in fair volume down to a particular age, stopping abruptly, while the graduated table is desired to reach younger ages. In such circumstances there are, in general, two courses open; first, the series may be continued graphically or by some process of extrapolation or, second, constant values of the ungraduated function corresponding to the last observed age, or sometimes hypothetical values, may be inserted as far

down as is necessary to enable the graduation to be carried to the lowest age required. Where some experience is available at these ages, it may be used as a guide for the insertion of hypothetical figures. Woolhouse used a process of finite differences, deducing a third difference from an assumed radix at age ten and the first and second differences at age seventeen, the lowest age reached by the application of his formula, completing the table down to age ten by adding differences in the usual way. Spencer and Hardy have advocated the use of constant average values for completing the series at the early ages and have obtained very satisfactory results in this way, but, of course, all of these processes are more or less arbitrary, and this should be remembered before giving undue weight to the graduated results at these ages. At the other end of the table it is usually the case that the data extend to the oldest age required, but, for some years, are very irregular and quite unsuitable for use without adjustment. Here there is great scope for the exercise of judgment in correcting accidental deviations, a good deal of which must be done by inspection and with the aid of standard tables. Having corrected all the glaring irregularities, it will be possible to apply the summation formula up to a fairly advanced age, but for the extreme ages special methods must again be resorted to. One very good way is to use a shorter summation formula, joining on the results obtained to the last values obtained by use of the principal formula. Sometimes it will be necessary to insert values from some standard table to supplement the facts at extreme ages, or to use average values. In graduating q_x Mr. Spencer has completed the table at the old ages by inserting unity at the limiting age and a hypothetical series of numbers, being powers of the constant c, at ages beyond. In doing this he treats the table of q_x as a series of numbers without reference to their meaning, and this, when the objects of graduation are remembered, seems perfectly legitimate and has the advantage of leading to a normal progression of the differences of the graduated figures.

Life, Term and Endowment Mortality Experience of the Provident Life and Trust Company of Philadelphia; with Especial Reference to Relative Vitality According to Age at Entry.

BY

MORRIS A. LINTON.

The following investigation of the mortality experience of The Provident Life and Trust Company was made for the purpose of studying the relative mortality under life, term and endowment policies and of obtaining facts bearing upon the relationship between the vitality of assured lives and the age at which they become assured. As the investigation progressed, additional features of interest were encountered and have been included in the results which are subdivided as follows:

I. Description of Data.
II. Life, Term and Endowment Experiences.
III. Relative Vitality according to Age at Entry.
IV. Supplement.
  (*a*) Construction of Ultimate Mortality Table based upon the Company's Experience.
  (*b*) Characteristics of the Provident Table.

I.

Description of Data.

The data involve the experience of the company from its foundation in 1865 to the end of 1911. Standard lives only have been accepted, and, with the exception of less than ten non-participating policies, all contracts have been upon an annual dividend basis. Policies becoming paid-up through non-payment of premium have been included. The company's experience under extended term insurance did not commence until 1912 and hence is not involved in the present investigation.

The total number of policies issued has been approximately 170,-

000 for a total of 486 millions of insurance. The total number of lives involved is approximately 120,000, of whom 93 per cent. are men and 7 per cent. women. The average amount of each policy is $2,866, whereas the average amount per life insured is about $4,050. The average number of policies per life insured is 1.41, so that, of the total number of policies issued, about 29 per cent. are "duplicates."

The approximate aggregate exposures are

1. By policies ..... 1.3 millions
2. By lives ..... 1.0 millions
3. By amounts ..... 3,792 millions

Throughout the investigation the calendar year method has been followed. Had the data been in the form to permit of it, the policy year method would have been employed, and a test by a standard select table would have been considered. Under the conditions, no attempt was made to apply select mortality rates to calendar year exposures.

The following tables show the distribution of the experience with reference to the kind of policy. The amounts and numbers of policies are given to the nearest million and nearest thousand, respectively. Such functions, however, as the average entry age, average duration, ratio of actual to expected deaths, etc., have been obtained from the unrounded figures.

TABLE I.

Kind.	Entrants.	Average Age at Entry.	Exposure.	Average Duration of Risk.	Actual Deaths.	Expected Deaths by American Table.	Ratio: Actual to Expected Deaths.
Amounts—Nearest Million.							
Life.....	66	40.2 yrs.	527	8.0 yrs.	8.5	10.9	.778
Term.....	109	36.9	421	3.9	2.7	4.6	.591
Endowment	311	34.1	2,844	9.1	22.6	35.1	.644
Total.....	486	35.6	3,792	7.8	33.8	50.6	.668
Policies—Nearest Thousand.							
Life.....	20	38.3 yrs.	158	8.0 yrs.	2.4	3.1	.748
Term.....	26	35.1	109	4.1	.6	1.1	.520
Endowment	124	32.5	1,055	8.5	7.5	12.3	.611
Total.....	170	33.6	1,322	7.8	10.5	16.5	.631

TABLE II.

Kind.	Entrants as in Table I.	Percentage of Entrants.		
		Dying.	Withdrawing.	Existing.
Amounts.				
Life	66	12.8%	31.4%	55.8%
Term	109	2.5	46.6	50.9
Endowment	311	7.3	40.1	52.6
Total	486	6.9	40.4	52.7
Policies.				
Life	20	12.0%	29.7%	58.3%
Term	26	2.3	46.8	50.9
Endowment	124	6.1	36.8	57.1
Total	170	6.2	37.5	56.3

TABLE III.

SHOWING PERCENTAGES, BY AMOUNTS, OF TOTAL ENTRANTS, DEATHS AND EXPOSURES, REPRESENTED BY LIFE, TERM AND ENDOWMENT CONTRACTS.

Kind.	Entrants.	Deaths.	Exposures.
Life	13.6%	25.1%	13.9 %
Term	22.4	8.0	11.11
Endowment	64.0	66.9	75.0
Total	100.0	100.0	100.0

The most striking feature of the experience is the preponderance of the endowment element. By amounts, the endowment entrants, deaths, and exposures are, respectively, 64 per cent., 67 per cent., and 75 per cent. of those under the total experience of the company. The most favorable ratio of actual to expected deaths by the American Table is found under the term experience, and the question is raised as to whether the low mortality experienced by the company is more largely to be attributed to the predominating form of contract or to the careful selection of risks.

It is interesting to note that the average duration of the risk has been greater for endowment than for life contracts. The average amount of each life, term and endowment policy has been $3,366, $4,138 and $2,516 respectively.

The ratios of actual to expected deaths by the American Table for the aggregate experience of the company have been obtained on the basis of policies, lives and amounts. These ratios are .631, .644,

and .668 respectively. The mortality by lives in the present instance lies between that by policies and that by amounts and is closer to the experience by policies.

The mortality (by lives) among women has, in the aggregate, been more favorable than that among men. By the American Table, the ratio of actual to expected deaths has been for men and women .647 and .602, respectively. The following table shows that the favorable experience among women arises from the exposures at attained ages above 35 or 40. As in previous experiences with assured lives, the incidence of the mortality among men differs from that among women—the men having greater vitality at the younger ages, the women at the older ages.

TABLE IV.

PERCENTAGE OF ACTUAL TO EXPECTED DEATHS, ON BASIS OF LIVES, BY AMERICAN TABLE.

Attained Ages.	Men.	Women.	Attained Ages.	Men.	Women.
Below 30	45%	55%	50–54	77%	66%
30–34	46	63	55–59	84	71
35–39	49	47	60–69	88	78
40–44	57	54	70 and over	0	59
45–49	64	51	All ages	65	60

II.

LIFE, TERM AND ENDOWMENT EXPERIENCES.

The well-known fact that the rate of mortality varies with the kind of policy, in conjunction with the fact that the aggregate experience of the Provident Life and Trust Company under term policies has been more favorable than the experience under any other form, renders an investigation as to kind of policy of especial interest.

In carrying out the investigation, two standards of reference were employed:

(1) The American Table, and (2) an ultimate table based upon the experience of the company. The method of construction of this table and the test of the graduation are set out in (*a*) of the supplement (page 304). It is based upon amounts and includes the experience of the company—excluding the first five years of exposure—from 1865 to the end of 1911. The graduation was ac-

complished by means of Spencer's twenty-one term summation formula.

The Provident Ultimate Table was constructed and employed in addition to the American Table, owing to a fear that the results shown by the latter table might be misleading. It is a well known fact that the American Table does not accurately represent the ultimate mortality experience of many American companies; that it is relatively much heavier than the actual experience at the younger than at the older ages. For example, the Provident ratio per cent. of the actual to the expected death loss by the American Table, excluding the first five years of exposure, for attained ages 30–39, is 55, and for ages 60–69, 93. Table XIV gives the ratio per cent. of the Provident ultimate q_x to the American q_x for quinquennial ages, and the general tendency of the ratio to increase up to age 65 is clearly seen. In a comparison of groups of policies by age at entry, it is essential that the standard of reference should adhere as closely as possible to the mean attained-age experience of the company after the effect of medical selection has largely disappeared; or at least should differ from that experience by a practically *constant* percentage. It will be seen from the series of ratios on page 305 that the $O^{M(5)}$, like the American Table, fails to fulfill this condition. An indication that the new M. A. Table would have been more satisfactory than either the American or $O^{M(5)}$ Table is evidenced by the series of ratios of the Provident ultimate q_x to that of the M. A. Table. It is realized that the two mortality rates are essentially different—the Provident rate referring to a single attained age, the M. A. rate to a group of ages—but the more constant value of the ratios is not without significance, and the employment of the M. A. Table would have been considered if the investigation had not been well under way when it was published. The Provident Table has been employed with confidence, as it does follow as closely as could be expected the attained-age experience of the company, excluding the first five years of exposure.

One of the interesting facts brought out by the investigation is that the incidence of the mortality according to years of exposure differs as to the kind of policy. This feature may be clearly seen in Table V.

TABLE V.

LIFE, TERM AND ENDOWMENT EXPERIENCES 1865–1911.

Grouped Years of Exposure.

Amounts and Policies.

Years of Exposure.	Percentage of Actual to Expected Deaths by American Table.			Percentage of Actual to Expected Deaths by Provident Table.			Actual Deaths (Amounts—Nearest Thousand).		
	Life.	Term.	Endt.	Life.	Term.	Endt.	Life.	Term.	Endt.
Amounts.									
1–5	66%	51%	45%	92%	84%	74%	1,853	1,389	5,015
6–10	78	68	62	100	103	93	1,707	807	5,608
11–15	81	59	75	98	81	102	1,336	276	4,931
16–20	95	110	78	111	137	100	1,180	233	3,663
21–25	96		91	111		111	901	5	2,178
26–30	80		89	92		103	574		839
31–end	70		106	80		117	921		364
1–end	78	59	64	97	92	93	8,472	2,709	22,598
6–end	82	71	73	98	101	100	6,620	1,320	17,583
11–end	84	74	80	98	99	104	4,913	514	11,975
16–end	85	105	84	98	131	104	3,576	238	7,044
21–end	81		92	92		110	2,396	5	3,382
1–20	77	59	61	99	92	90	6,076	2,704	19,216
Policies.									
1–5	60	43	45	86	72	75	478	285	1,872
6–10	71	64	59	93	98	91	439	195	1,916
11–15	78	59	70	96	83	99	359	72	1,570
16–20	92	85	76	110	109	99	320	44	1,142
21–25	93		85	108		105	261	1	662
26–30	83		87	95		101	191		260
31–end	75		91	85		101	306		96
1–end	75	52	61	94	82	90	2,354	597	7,518
6–end	80	64	69	97	94	97	1,876	312	5,646
11–end	83	66	76	98	89	100	1,437	117	3,730
16–end	85	81	80	98	103	101	1,078	45	2,160
21–end	82		86	94		104	758	1	1,018
1–20	72	52	58	94	82	89	1,596	596	6,500

Considering first the ratios for the life and endowment experience, we notice apparent contradictions between the ratios by the American and by the Provident Tables. These are largely owing to the fact that the endowment experience is more confined to those ages at which the American Table greatly exceeds the actual experience. The average age at entry is about six years lower for endowment than for life policies, and this difference has an important effect upon the relations between the two sets of ratios. Consider, for

example, the following series of ratios taken from the "amounts" section of Table V.

Years of Exposure.	American.		Provident.	
	Life.	Endowment.	Life.	Endowment.
All years	78	64%	97	93%
Excluding 1st 5 yrs.	82	73	98	100
Excluding 1st 10 yrs.	84	80	98	104
Excluding 1st 15 yrs.	85	84	98	104
Excluding 1st 20 yrs.	81	92	92	110

We note that, as usual, the aggregate experience is more favorable for the endowments; though the difference is exaggerated by the American Table. Furthermore, we note that, in spite of the fact that by the American Table the endowment ratio is inferior to the life ratio only for the group excluding the first twenty years of exposure, all of the endowment ratios by the Provident Table, with the exception of the aggregate ratio, are inferior to those of the life class. Analogous remarks apply to the experience by policies. For reasons stated above, I believe that it is the Provident Table which shows the truer results.

The ratios just considered exhibit the difference in the incidence of the mortality for the two kinds of policy. In the earlier years the endowments have made the better showing, but as the duration has increased they have deteriorated and the life experience has been more favorable.

This is also evidenced by the ratios for the five year groups of years of exposure. By both tables the endowments are superior for the first ten years. After ten years the Provident Table indicates a tendency toward an equality which the American Table does not reveal so clearly. However, after twenty-five years both series of ratios show that the life is more favorable than the endowment experience.

The fact that Table V has reference to all ages at entry and that on the endowment form the long durations of 25 years and over are confined more largely to the younger ages at entry, raises the question as to whether the divergence in the two experiences would be evidenced by data for the younger ages at entry only. The following figures are the result of an investigation of this point. They

are the ratios per cent. by amounts, of actual to expected deaths by the Provident Table for ages at entry 20 to 34.

Years of Exposure.	Life.	Endowment.
1–40	107%	97%
6–40	106	105
11–40	100	111
16–40	105	112
21–40	101	118
26–40	93	114

Although these ratios differ in magnitude from the corresponding ratios of Table V, the relationship between the life and endowment experiences is substantially the same and leads to the same conclusion: that there has been a marked divergence in the incidence of the mortality of the two classes.

As an explanation of the phenomenon, initial self-selection on the part of the insured could hardly be considered. Three possible explanations are suggested.

1. That the deterioration of the endowments in the later years of duration is due to the fact that throughout its history the company has adopted the common practice of limiting lives with slightly under average prospects of longevity, to the endowment form. If this be the true explanation, we may congratulate the medical director upon having caused the death strain to fall where it would be least felt.

2. That it is due to a difference in the lapse rates of the two classes of policies. To as great an extent as the data would permit I have investigated the lapse rate of the Provident and have found that for the first ten years of duration (calendar year method) the endowments have experienced a much lighter rate of discontinuance. The classification of the data is such that under the heading "Retirants" have been included all discontinuances, whether voluntary or involuntary, so that, after the tenth calendar year, maturities render impracticable the determination of the rate of voluntary withdrawal of the endowments. As a rough indication of the difference between the two lapse rates there is given below the ratio per thousand (by policies) of retirants to exposures for nine individual years of exposure. The ratios are based upon the Provident experience, 1865–1911.

Year of Exposure.	Life.	Endowment.
2	123	109
3	64	56
4	45	40
5	36	34
6	40	30
7	36	26
8	27	22
9	30	20
10	21	16

I have also investigated the ratio of retirants to exposures for decennial age-at-entry groups and have found that the smaller groups exhibit the same general characteristics as the group for all ages at entry, namely, that the life policies have experienced the heavier lapse rate. Therefore, if the superiority of the life class at the longer duration has been caused by withdrawals, the inference is that the withdrawals have been relatively unhealthy lives. The relation between the two lapse rates at the longer durations is not known, though in view of the larger premium and larger cash value of the endowments, it might be surmised that their rate of lapse would exceed that of the life class.

With the exception of the Connecticut Mutual experience (1846–1878) and of deferred dividend experiences, I have been able to find no data bearing upon the relationship between the lapse rates of life and endowment policies. The Connecticut Mutual experience indicates a heavier lapse rate among the endowments; the deferred dividend experiences a lighter one. As a matter of interest I have calculated from the published British Offices' 1863–93 experience the rate per thousand of "non-renewal" (Todhunter, *J. I. A.*, XXXIII, 273) for the "New Assurances," life and endowment, males. The results are as follows:

Policy Year.	Life (Participating).	Endowment.
1	70	73
2	50	46
3	38	36
4	30	28
5	26	25
6	22	21
7	19	19
8	17	17
9	15	16
10	15	15
11–15	12	13
16–20	10	10
21–30	7	8

Though not so markedly as in the Provident experience, the endowments exhibit, in general, a lighter lapse rate in the earlier years of duration. In the later years the life and endowment rates are nearly the same, with a slight tendency on the part of the endowment rate to become heavier.

3. There may be a difference between the mortality on endowments for twenty years or less and on those for a longer period. If the long endowments should throughout be inferior to the short endowments and to the life class, then an apparent deterioration similar to that actually observed would be produced by the maturing of the policies for the shorter period.

Which of these explanations is the true one it is at present impossible to say. They may all have had an important bearing upon the subject. The problem is complex and interesting and invites further investigation.

The ratios in Table V represent the experience for all ages at entry. In Tables IX, X, XI and XII the results have been subdivided into decennial age groups and according to grouped years of exposure.

In spite of the inferiority at the longer durations, we find, as in previous experiences, that during the earlier years the endowment has, in general, been more favorable than the life mortality. The selection by the medical department is more exacting for the life than for the endowment entrants, and it would seem that self-selection on the part of the applicant had been responsible for their more favorable experience during the early years of insurance. Under "self-selection" is included not only a possible intuitive conviction of longevity, but also the impulse on the part of the thrifty man to prefer a certain type of insurance. In view of the great influence wielded by the agent in determining the kind of policy which the applicant selects, it is difficult from *a priori* reasoning to attribute great importance to initial self-selection on the part of the insured. On what other hypothesis, however, is it possible to account for the light mortality experienced in the early years of insurance; for the fact that this light mortality becomes apparent in the first policy year? And this in spite of the fact that the medical selection of the life entrants is more exacting.

In the present experience the relative deterioration among the endowments of long durations is an additional feature of interest. However strong the initial self-selection in favor of the endowments,

there has been some influence which, with increasing duration, has overcome it and has rendered the life mortality the more favorable.

The experience under term policies has been relatively favorable. A large proportion of this business is renewable without medical examination and on that account the utmost care is taken to render the initial examination thorough. The results in Table V show that, considering the adverse selection that has at times been evidenced by term mortality, the efforts have been effective; though, owing both to the short duration and the young age at entry in comparison with the life class, and to the short duration in comparison with the endowments, the ratios by the American Table unduly favor the term business.

The comparisons by amounts and by policies display interesting divergencies. By the Provident Table, the ratios per cent. of actual to expected deaths for the first twenty years are as follows:

	Amounts.	Policies.
Life	99%	94%
Term	92	82
Endowment	90	89

As in other American companies, the mortality by policies has been more favorable than by amounts. In the present instance the greatest difference is found under the term, the least under the endowment experience. The evidence is therefore, that relatively heavy mortality has been experienced among policyholders who carry policies of more than average amount, and that this characteristic has been most pronounced in the term class. The above table shows that the term ratio is in both instances superior to that for life. By amounts the term is slightly inferior to the endowment experience, but by policies it is markedly superior to it.

One question that is of vital interest in connection with renewable term policies is the selection which the insured may exercise at the renewal periods. In the present experience the renewable and non-renewable term data have not been separated. Of the total term insurance issued about 15 per cent. has been non-renewable. Neither has the experience been subdivided as to kind of term policy, i. e., into five, ten, fifteen and twenty year classes. However, the majority of cases has been on the ten-year renewable form, and this class has been the controlling factor in the experience. With

the object of throwing light upon the question of selection at the renewal periods, the experience as to each individual year of exposure (calendar year method) has been investigated, with the results shown in Table VI.

TABLE VI.

TERM EXPERIENCE 1890–1911.

By Years of Exposure.

Amounts and Policies.

Year of Exposure.	Actual Deaths.		Percentage of Actual to Expected Deaths by American Table.		Percentage of Actual to Expected Deaths by Provident Table.	
	Amounts (Nearest Thousand.)	Policies.	Amounts.	Policies.	Amounts.	Policies.
1	254	44	47%	35%	79%	60%
2	285	77	35	39	58	67
3	230	58	40	41	66	69
4	328	60	75	54	122	89
5	292	46	84	51	136	84
6	190	47	65	62	103	99
7	161	35	62	52	96	81
8	167	45	72	75	109	115
9	183	40	87	74	127	110
10	106	28	56	57	80	84
11	40	14	28	37	39	53
12	48	15	48	57	67	81
13	45	16	53	73	72	102
14	46	16	63	83	85	114
15	98	11	149	65	196	87
16	41	10	70	69	90	92
17	59	11	114	88	144	113
18	59	11	140	105	174	133
19	41	6	119	72	146	90
20	33	6	130	99	156	121
1–20	2,704	596	59	52	92	82
6–20	1,315	311	71	65	102	95
11–20	509	116	75	67	100	91
16–20	233	44	110	85	137	109
1–5	1,389	285	51	43	84	72
6–10	807	195	68	64	103	98
11–15	276	72	59	59	81	83
16–20	233	44	110	85	137	109

If there had been adverse selection against the company, one would expect to find evidence of it in the ratios for the two years succeeding the fifth, tenth and fifteenth years of exposure. It will be recalled that the calendar year method divides the experience into years that end, on the average, in the middle of a policy year.

Thus the eleventh year of exposure extends, on the average, from the end of $9\frac{1}{2}$ to the end of $10\frac{1}{2}$ policy years. Therefore, adverse selection on the ten-year term policies, if it had existed, should begin to evidence itself in the eleventh and twelfth years. It appears, however, that the ratios for these two years are more favorable than the average, and the ratios for the eleventh year are, with one exception, the lowest of the series. It would be a mistake to generalize on these conclusions; to allow them to close our eyes to the opportunities for adverse selection which the renewal periods afford. It would seem, however, that if adverse selection has been exercised it has not been of the type which was followed by an immediately unfavorable experience. The relatively highest mortality has been experienced during the years of exposure fifteen and over. Moreover, in this section the experience has been unduly heavier by amounts than by policies. The average amount of each policy becoming a claim during the years sixteen to twenty is $5,291, whereas for the entire term experience the average is $4,537. It would, therefore, seem that in the later years, the adverse mortality had been experienced upon the larger policies. It is also a matter of interest that the average amount of the policy becoming a claim in the first year, i. e., becoming a claim in the calendar year of issue, is $5,782, whereas for the second and third years the values are $3,695 and $3,964 respectively.

III.

Relative Vitality According to Age at Entry.

In making an investigation into the relationship between the vitality of assured lives and the age at which they become assured, the problem is to throw into relief the experiences of the several ages at entry, throughout those years of life that are common to the given ages. It is not sufficient for this purpose to compare the ratio of actual to expected deaths for the entire experience under one group of ages at entry with a similar ratio for another group of ages. This latter comparison is of value in comparing the experiences of different classes under the same ages at entry, in passing upon the validity of the standard of reference as a basis for premium calculations, but it does not refer exclusively to the years of life that are common to the given ages at entry. The fact that it is necessary that the experiences under two age groups should

cover the same span of life in order that their relative vitality may be determined, imposes limitations upon the possibility of achieving the end in view. For, in order that age at entry 25, for example, be contrasted with age at entry 45, twenty years of duration must have elapsed before the latter age comes on the books, and when that does occur, the former is twenty years away from medical selection, whereas the latter has just come from the doctor's office. Before any valid comparison can be instituted, the effects of selection for age 45 must be allowed to wear off, i. e., we must wait another five or ten years and allow age 25 to complete twenty-five or thirty years of duration before the two can be thrown into relevant contrast. The subject is further complicated by the improvement in vitality that is continually taking place in the community as a whole.

In general, the subject is largely one of academic interest only. A knowledge of the relative vitality of ages at entry from 25 to 50 would mean little to us in our daily work. The only ages that interest us in this connection are those at the two extremes; those below 25 and those over 50. With regard to these two groups of ages more definite information is desirable.

The present investigation covers the period 1865–1911, so that the maximum duration that may be practicably employed in the comparisons is about forty years. For purpose of comparison, the ages at entry were taken in five-year groups, each group being designated by its central age. No comparison was instituted before at least five years had elapsed since medical examination. Thus in comparing groups 22 and 27 through a period of thirty years, the years of exposure 11–40 and 6–35, respectively, were taken, so that the data cover the span of life which we may designate 32–61. It will be understood that we are thus comparing, by means of a single ratio, five pairs of ages at entry, namely, ages 20 and 25 at entry, through attained ages 30–59, ages 21 and 26 through ages 31–60, and so on up to 24 and 29 through ages 34–63. There is one exception to the general scheme as just outlined, namely, that the group designated $62\frac{1}{2}$ refers to the six ages at entry 60–65 inclusive. To have been absolutely consistent with the other groups, age 65 should have been excluded. But, owing to the fact that the data at these old ages are meager, and to the fact that age 65 is the oldest age at which applicants have been regularly accepted, it was thought best to waive the consistency and to include it.

The maximum duration of forty years refers only to life policies;

the maximum that is practicable under endowments is thirty-five years,—and this duration refers only to the younger ages at entry,—so that the maximum comparison span is reduced to twenty-five years. Owing to the fact that the term experience is so largely concentrated at the shorter durations and that the company has been issuing term insurance since 1890 only, the comparison for the purpose of determining the relative vitality of the several ages at entry cannot be satisfactorily carried out.

In connection with the endowment and life experiences, a large number of comparisons were instituted and ratios obtained, the inclusion of which in the *Transactions* would contribute little of value. Therefore, the following tables contain merely those results that are essential in solving the given problem. Since the interest centers chiefly around the two extremes of the ages at entry, it is the data for these ages that are given the greatest attention. In drawing conclusions based upon the results, the ratios under the Provident Table have been the most helpful. By the American Table nearly all of the ratios are less than 100; by the Provident Table, which is a mean of the entire ultimate experience of the company, the ratios cluster about 100 and their significance is the more readily grasped.

The endowment experience is considered first, and the results pertaining to it are set out in Table VII.

TABLE VII.

ENDOWMENT EXPERIENCE 1865–1911.

Comparison of Groups of Five Ages at Entry as They Pass through the Same Attained Ages.

Ages at Entry.	Years of Exposure.	Ages Passed Through.	Percentage of Actual to Expected Deaths by American Table.		Percentage of Actual to Expected Deaths by Provident Table.		Actual Deaths.	
			Amts.	Policies.	Amts.	Policies.	Amounts (Nearest Thousand).	Policies.
25 Years of Life.								
22	11–35	32–56	70%	69%	117%	117%	784	307
27	6–30		63	60	103	100	2,322	883
27	11–35	37–61	69	65	107	102	1,730	605
32	6–30		70	66	106	101	3,332	1,110
32	11–35	42–66	82	75	114	105	2,628	819
37	6–30		68	67	93	92	3,034	982

TABLE VII. (*continued*).

Ages at Entry.	Years of Exposure.	Ages Passed Through.	Percentage of Actual to Expected Deaths by American Table.		Percentage of Actual to Expected Deaths by Provident Table.		Actual Deaths.	
			Amts.	Policies.	Amts.	Policies.	Amounts (Nearest Thousand).	Policies.
20 Years of Life.								
22	16–35	37–56	75	78	121	127	508	195
27	11–30		68	64	107	102	1,619	576
32	6–25		67	64	104	100	2,997	1,011
27	16–35	42–61	76	70	108	101	1,188	391
32	11–30		81	75	113	105	2,537	794
37	6–25		68	67	94	92	2,905	941
32	16–35	47–66	90	83	114	106	1,794	546
37	11–30		74	73	92	91	2,122	678
42	6–25		79	75	97	93	2,688	791
15 Years of Life.								
22	21–35	42–56	85	86	124	127	307	105
27	16–30		74	69	108	101	1,078	362
32	11–25		78	72	112	104	2,202	695
37	6–20		63	62	90	89	2,346	768
32	21–35	52–66	108	96	126	112	1,108	319
37	16–30		82	80	95	92	1,373	426
42	11–25		84	83	97	96	1,769	532
47	6–20		80	83	92	95	2,042	583
42	16–30	57–71	88	87	97	97	991	294
47	11–25		85	87	93	96	1,386	374
52	6–20		96	90	106	99	1,496	393
52	11–25	62–76	114	104	124	113	1,002	247
57	6–20		90	81	99	89	605	172
57	11–25	67–81	84	76	95	86	252	72
62½	6–20		101	89	115	101	307	77

TABLE VII. (*continued*).

Ages at Entry.	Years of Exposure.	Ages Passed Through.	Percentage of Actual to Expected Deaths by American Table.		Percentage of Actual to Expected Deaths by Provident Table.		Actual Deaths.	
			Amts.	Policies.	Amts.	Policies.	Amounts (Nearest Thousand).	Policies.
10 Years of Life.								
22	26–35		108%	104%	142%	137%	173	52
27	21–30		82	76	108	101	618	199
32	16–25	47–56	85	79	112	104	1,368	422
37	11–20		67	66	87	87	1,434	464
42	6–15		74	70	97	91	1,752	516
32	26–35		110	102	122	113	426	124
37	21–30		95	93	106	103	688	214
42	16–25	57–66	88	87	98	96	936	275
47	11–20		85	89	94	99	1,253	348
52	6–15		96	88	106	97	1,225	320
42	21–30		78	81	84	88	277	89
47	16–25	62–71	91	87	99	94	697	164
52	11–20		114	104	124	113	963	239
57	6–15		92	81	100	88	553	153
52	16–25		95	99	105	109	310	81
57	11–20	67–76	85	76	96	86	251	71
62½	6–15		100	87	115	98	293	71
57	16–25	72–81	74	85	91	104	52	20
62½	11–20		129	108	158	132	122	30

In making the following comparisons, it will be remembered that the younger the age in a group of ages at entry, the farther away it is from medical selection and the more likely it is to make a slightly poorer showing than those which are older. On the other hand, if the younger age makes the better showing, it does so in spite of the fact that the others have been more recently selected.

From the section of this table representing the comparison through twenty-five years of life, we obtain information regarding the younger ages at entry. In comparing age groups 22 and 27, we find that 22 has possessed distinctly lower vitality than 27. Not only is this evidenced in the comparison through twenty-five years, but also in all of the comparisons in which it appears. The difference is well sustained the farther we get from medical selection. As we investigate the next two groups, 27 and 32, in conjunction with 22, we note that no one of the three has made a good showing

in comparison with groups 37 and 42, and we arrive at the general conclusion that the ages at entry below 35 have possessed distinctly lower vitality than those from 35 to 50, and that those below 25 have been the most inferior of all. As we approach the older ages, we note that group 52 reveals a relatively unfavorable experience. This is especially noticeable in comparison with groups 42 and 47. Group 57 seems to have been an improvement upon 52, but the experience under 62½ has been distinctly unfavorable.

TABLE VIII.

LIFE EXPERIENCE 1865–1911.

Comparison of Groups of Five Ages at Entry as They Pass through the Same Attained Ages.

Ages at Entry.	Years of Exposure.	Ages Passed Through.	Percentage of Actual to Expected Deaths by American Table.		Percentage of Actual to Expected Deaths by Provident Table		Actual Deaths.	
			Amts.	Policies.	Amts.	Policies.	Amounts (Nearest Thousand).	Policies.
30 Years of Life.								
22	11–40	32–61	69%	68%	99%	100%	62	24
27	6–35		67	64	103	98	288	102
27	11–40	37–66	73	77	100	105	259	99
32	6–35		74	78	104	109	489	166
32	11–40	42–71	79	86	100	108	440	154
37	6–35		86	79	112	102	684	197
37	11–40	47–76	93	82	112	98	593	170
42	6–35		87	81	105	97	890	233
42	11–40	52–81	92	82	105	94	775	204
47	6–35		85	85	98	98	1,019	268
47	11–40	57–86	82	87	93	99	778	222
52	6–35		79	79	89	89	1,055	291

TABLE VIII. (*continued*).

Ages at Entry.	Years of Exposure.	Ages Passed Through.	Percentage of Actual to Expected Deaths by American Table.		Percentage of Actual to Expected Deaths by Provident Table.		Actual Deaths.	
			Amts.	Policies.	Amts.	Policies.	Amounts (Nearest Thousand).	Policies.
20 Years of Life.								
22	21–40	42–61	79	80	100	101	42	15
27	16–35		85	85	113	112	174	63
32	11–30		80	87	107	117	307	106
37	6–25		87	78	120	107	530	145
32	21–40	52–71	88	93	99	105	290	102
37	16–35		92	86	104	97	376	115
42	11–30		101	88	115	100	649	160
47	6–25		87	87	99	99	827	209
42	21–40	62–81	89	79	100	89	424	119
47	16–35		81	85	91	96	507	147
52	11–30		84	84	94	95	768	213
57	6–25		89	87	100	98	825	198
47	21–40	67–86	74	82	85	95	362	114
52	16–35		86	86	99	100	589	171
57	11–30		93	91	107	105	617	151
62½	6–25		105	100	122	116	636	161
52	21–40	72–91	82	84	97	98	371	116
57	16–35		80	86	93	100	361	98
62½	11–30		97	103	114	121	315	93

TABLE VIII. (*continued*).

Ages at Entry.	Years of Exposure.	Ages Passed Through.	Percentage of Actual to Expected Deaths by American Table.		Percentage of Actual to Expected Deaths by Provident Table.		Actual Deaths.	
			Amts.	Policies.	Amts.	Policies.	Amounts (Nearest Thousand).	Policies.
15 Years of Life.								
22	26–40		82%	69%	98%	82%	34	10
27	21–35		91	91	111	110	128	47
32	16–30	47–61	94	101	115	124	239	82
37	11–25		105	88	131	109	395	102
42	6–20		81	77	101	96	513	129
32	26–40		90	92	99	101	225	77
37	21–35		91	84	101	93	264	81
42	16–30	57–71	110	93	121	103	487	120
47	11–25		87	91	96	100	548	147
52	6–20		76	76	84	84	700	184
42	26–40		72	71	84	82	239	77
47	21–35		77	81	89	94	323	98
52	16–30	67–81	88	89	101	103	533	154
57	11–25		94	91	108	105	552	133
62½	6–20		107	101	124	118	613	150
47	26–40		73	82	86	96	230	75
52	21–35	72–86	86	84	101	99	355	107
57	16–30		89	91	104	107	348	92
62½	11–25		99	108	117	127	312	91
52	26–40		80	80	91	91	198	64
57	21–35	77–91	77	80	87	90	202	53
62½	16–30		138	101	157	114	175	41

The life ratios which are given in Table VIII differ at the younger ages in an interesting manner from those of the endowment experience. In the first place, it will be noticed that the majority of the ratios indicate a relatively high vitality for group 22. The data upon which these life ratios are based are meager, but it is believed that they are sufficient to indicate that, until the late thirties and early forties are reached, there has been a general tendency toward lower vitality as the age at entry increased. After middle life an improvement sets in, and we find that entrants from age 45 to 55 have made a good showing. Above age 55 a deterioration again occurs and culminates in the oldest age group, 62½.

It is realized that the above comparison has its limitations; that the foundation will not bear an elaborate superstructure. But it is

believed that the results do tend to throw light upon the subject they were intended to illumine. Comparing the ratios for the life and endowment experiences, it would seem that, whereas the life entrants below age 25 had possessed somewhat greater vitality than those which were older, the endowment entrants below age 25 had been of relatively poor grade. The life experience indicates a deterioration as middle life is approached and thereafter an improvement up to age 55. The endowments, on the other hand, improve from age 25 to age 50 at entry and then exhibit a high mortality for entrants in the early fifties. At the old ages, both classes evidence a low vitality which reaches its maximum among entrants above 60. It has been repeatedly pointed out that the legitimate need for insurance protection becomes less urgent at the old ages and that great care is needed to guard against adverse selection. The Provident experience bears additional testimony to the soundness of this principle.

The differences exhibited by the life and endowment experiences are explained by the fact that young entrants with defective family history are accepted, if at all, on the endowment plan. The Provident agents are aware of this fact, and where there is a doubt as to the character of the family record, they are apt to advise the form of insurance that experience has shown will be granted. Therefore, in addition to the customary limitation by the medical director, there is a selection made by the agent before ever the application reaches the office. The result has been that among the younger endowment entrants has been included an undue proportion of slightly inferior lives.

To avoid possible misapprehension, it may be pointed out that, although in summarizing the results of the investigation we pass successively from the youngest to the oldest age at entry, we do not thereby imply that the two extremes of age have been contrasted with each other. The comparison for any given age involves only the ages that immediately surround it, and as we proceed from group to group the "environment" with which any particular age is contrasted is continually changing.

I cannot but realize that the results of the investigation are in some cases based upon meager data and on that account display contradictions. In forming the conclusions it has been necessary to attempt to interpret the undercurrent beneath the irregularities. The conclusions cannot be absolute; they simply indicate general tendencies.

In conclusion, it will be of interest to recall the facts concerning the relative vitality of entrants of different ages, that were revealed by the British Offices' experience. We find definite reference to the subject on pages 146–7 of the volume on "Principles and Methods," where the $O^{[M]}$ experience is discussed. The method there adopted was that of comparing values of the function $e_{[x-n]+n}$ for different values of n and x, such that the attained ages entering into the comparison were the same. This method of comparison is satisfactory where the data for each age at entry extend to the limit of life. It is unsuitable for an experience of limited duration or one involving endowment policies. The conclusion is reached that "the age at entry in certain cases materially affects the mortality throughout the whole of life. For example, lives entering between 20 and 24, and in still more marked manner those entering between 25 and 29, show better rates of mortality at all ages passed through than the average. Twenty-five years after entry the average expectation of life $e_{[27]+25}(=19.05)$ is distinctly higher than that of a life of corresponding present age having entered only five years previously $e_{[47]+5}(=18.73)$." In commenting upon the phenomenon, Mr. Todhunter (*J. I. A.*, XXXVIII, 360) suggests that "the greater importance attached to family history in the selection of entrants at the younger ages might conceivably render them superior in later life to entrants at older ages, and it is to be remembered that the younger entrants who come under observation at the older attained ages in the $O^{[M]}$ experience were selected many years ago when family history was perhaps more considered than it is now, but in view of the fact that all entrants up to age 44 appear ultimately to become better lives than entrants at subsequent ages, it certainly seems as if the true explanation of the feature in question is to be found in some cause of a more general nature than the superiority of the younger entrants." He also suggests that withdrawals may have been a factor in producing the observed characteristics.

Regarding life entrants below 45 years of age, the results of the present investigation agree in large measure with those of the $O^{[M]}$ experience, in that both reveal an apparent deterioration in the vitality as the age at entry increases. With reference to the older ages at entry, it does not appear from the table of expectations of the $O^{[M]}$ experience, that they were of relatively low vitality.

The British Offices' experience is based upon lives and it has been

for want of a better expression that the word "vitality" has been employed in connection with an investigation by policies and amounts.

I have found no reference in the British Offices' experience to the relative vitality of endowment entrants according to age at entry. Mr. Hunter (*T. A. S. A.*, VII, 117) reviewed the experience, employing as the standard of reference a modification of the American Table. He draws the following conclusion with regard to endowments: "The mortality under endowment insurances for decennial groups of ages at entry, exhibits an increasing ratio of the standard, i. e., 69 per cent., 89 per cent., 96 per cent., and 121 per cent. for the four decennial age-periods from 20 to 59, showing that young lives, on which endowment insurance is asked for, are of the very best." Mr. Hunter's ratios do not refer exclusively to the experience of the different age groups as they pass through the same attained ages, and therefore do not attack the problem of relative vitality from the exact point of view of the present investigation. However, the series of ratios are such that it is hardly possible that, at the younger ages, the vitality of the entrants would display the characteristics brought out by the present investigation, namely, that the vitality of entrants below 35 had been relatively low.

Bearing in mind the conclusions that have been drawn from Tables VII and VIII, it is interesting to examine the ratios of Tables X and XII. We note that the Provident ratios, shown by the latter tables, do indicate just such characteristics as were revealed by a study of the former ones. The American ratios, on the other hand, are positively misleading, and they justify the labor necessary to construct and apply the Provident Table. If the standard of reference is not true to the mean ultimate experience of the class that is being investigated, comparisons of aggregate ratios for different ages at entry may reflect, not the intrinsic characteristics of the several ages, but simply certain characteristics originating in the standard itself.

The fact that the Provident ratios of Tables X and XII lead to the same conclusions regarding the relative vitality of entrants as do the ratios of Tables VII and VIII, suggest the probability that the term ratios of the former tables are a correct indication of the relative vitality of the term entrants. On this hypothesis, the relationships between the decennial age groups are probably in accord

with the following ratios taken from the sections relating to all years of exposure.

Ages at Entry.	Amounts.	Policies.
Below 30	105%	87%
30–39	88	75
40–49	85	80
50–59	120	110

TABLE IX.

LIFE, TERM AND ENDOWMENT EXPERIENCES 1865–1911.

Grouped Ages at Entry, Grouped Years of Exposure.

Amounts.

Ages at Entry.	Percentage of Actual to Expected Deaths by American Table.			Percentage of Actual to Expected Deaths by Provident Table.		
	Life.	Term-	Endowment.	Life.	Term.	Endowment.
Years of Exposure 1–5.						
Below 30	39%	57%	45%	73%	105%	84%
30–39	57	43	37	103	78	67
40–49	66	45	48	99	69	72
50–59	83	111	64	96	132	75
60 and over	67	—	65	72	—	71
All ages	66	51	45	92	84	74
Years of Exposure 6–10.						
Below 30	67	67	52	121	120	95
30–39	66	51	53	113	88	92
40–49	75	88	71	97	117	93
50–59	75	87	81	82	97	90
60 and over	110	—	93	125	—	104
All ages	78	68	62	100	103	93
Years of Exposure 11–20.						
Below 30	53	49	61	92	86	108
30–39	87	82	68	121	117	96
40–49	94	70	85	105	81	97
50–59	86	106	106	97	119	117
60 and over	91	—	128	106	—	157
All ages	87	75	76	104	100	101
Years of Exposure 21–End.						
Below 30	82		83	97		111
30–39	85		102	96		116
40–49	82		78	93		85
50–59	76		95	87		116
60 and over	61		—	70		—
All ages	81		92	92		110

TABLE X.

LIFE, TERM AND ENDOWMENT EXPERIENCES 1865–1911.

Grouped Ages at Entry, Grouped Years of Exposure.

Amounts.

Ages at Entry.	Percentage of Actual to Expected Deaths by American Table.			Percentage of Actual to Expected Deaths by Provident Table.		
	Life.	Term.	Endowment.	Life.	Term.	Endowment.
All Years of Exposure.						
Below 30	61%	58%	57%	94%	105%	98%
30–39	74	51	60	106	88	91
40–49	80	60	70	98	85	89
50–59	80	104	84	91	120	94
60 and over	86	—	88	97	—	99
All ages	78	59	64	97	92	93
Excluding First Five Years of Exposure.						
Below 30	70	59	63	101	106	104
30–39	81	63	69	106	100	100
40–49	84	81	79	98	102	94
50–59	80	93	94	90	103	104
60 and over	97	—	106	112	—	122
All ages	82	71	73	98	101	100
Excluding First Ten Years of Exposure.						
Below 30	71	47	69	96	82	109
30–39	86	81	78	105	115	103
40–49	87	69	84	99	79	95
50–59	82	106	106	93	118	117
60 and over	86	—	136	101	—	166
All ages	84	74	80	98	99	104

TABLE XI.

LIFE, TERM AND ENDOWMENT EXPERIENCES 1865–1911.

Grouped Ages at Entry, Grouped Years of Exposure.

Policies.

Ages at Entry.	Percentage of Actual to Expected Deaths by American Table.			Percentage of Actual to Expected Deaths by Provident Table.		
	Life.	Term.	Endowment.	Life.	Term.	Endowment.
Years of Exposure 1–5.						
Below 30	40%	42%	44%	74%	78%	82%
30–39	54	34	40	97	61	71
40–49	71	47	49	107	72	75
50–59	67	90	57	77	107	66
60 and over	64	—	81	70	—	88
All ages	60	43	45	86	72	75
Years of Exposure 6–10.						
Below 30	54	67	52	98	120	94
30–39	63	52	53	108	90	93
40–49	71	73	68	92	96	89
50–59	74	89	78	81	99	86
60 and over	89	—	83	101	—	92
All ages	71	64	59	93	98	91
Years of Exposure 11–20.						
Below 30	54	40	60	96	71	106
30–39	82	66	66	114	95	94
40–49	91	69	86	102	79	98
50–59	86	132	96	96	145	105
60 and over	91	—	109	106	—	134
All ages	84	67	73	102	91	99
Years of Exposure 21–End.						
Below 30	87		80	103		107
30–39	85		93	96		106
40–49	81		77	92		83
50–59	80		87	92		105
60 and over	72		—	82		—
All ages	82		86	94		104

TABLE XII.

LIFE, TERM AND ENDOWMENT EXPERIENCES 1865–1911.

Grouped Ages at Entry, Grouped Years of Exposure.

Policies.

Ages at Entry.	Percentage of Actual to Expected Deaths by American Table.			Percentage of Actual to Expected Deaths by Provident Table.		
	Life.	Term.	Endowment.	Life.	Term.	Endowment.
All Years of Exposure.						
Below 30	59%	48%	54%	93%	87%	95%
30–39	72	43	58	102	75	90
40–49	79	57	69	97	80	89
50–59	78	96	76	88	110	86
60 and over	80	—	88	91	—	98
All ages	75	52	61	94	82	90
Excluding First Five Years of Exposure.						
Below 30	69	57	61	100	101	102
30–39	79	57	66	103	91	96
40–49	82	71	78	95	89	94
50–59	81	103	87	91	114	96
60 and over	88	—	93	101	—	106
All ages	80	64	69	97	94	97
Excluding First Ten Years of Exposure.						
Below 30	74	39	66	101	69	107
30–39	84	66	74	102	94	98
40–49	85	67	85	96	78	96
50–59	83	130	96	94	144	105
60 and over	87	—	115	101	—	140
All ages	83	66	76	98	89	100

IV.

Supplement.

(*a*) *Construction of Provident Table.*

The Provident Table is based upon the total experience of the company prior to 1912, excluding the first five ($4\frac{1}{2}$) years of assurance. It is based upon sums assured. The graduation was performed by means of Spencer's twenty-one term summation formula (*J. I. A.*, XXXVIII, 338), a formula, as is well known, possessing remarkable smoothing power. An experiment was made to determine the function best adapted to graduation. The formula was applied (1) to the rates of mortality deduced directly from the exposures and deaths, and (2) to the ratio of actual to expected deaths by the American Table. The two series of graduated rates thus obtained were applied to the exposures, with the result that the series deduced from the graduated ratio of actual to expected deaths was adopted. The deviations and accumulated deviations by the first series were less satisfactory than those of the second series. The application of the rates adopted yield the results set out in the following table. The figures are given to the nearest $1,000 omitting the ciphers.

TABLE XIII.

Attained Ages.	Exposed to Risk Excluding First Five Years.	Actual Deaths.	Expected Deaths Provident Ultimate Table.	Deviation.	Accumulated Deviation.	Accumulated Actual Deaths.
Below 25	5,088	24	21	− 3	− 3	24
25–29	58,176	269	257	− 12	− 15	293
30–34	183,087	878	884	6	− 9	1,171
35–39	324,171	1,620	1,627	7	− 2	2,791
40–44	411,080	2,405	2,410	5	3	5,196
45–49	413,484	3,195	3,287	92	95	8,391
50–54	344,484	4,198	4,080	−118	− 23	12,589
55–59	239,671	4,387	4,351	− 36	− 59	16,976
60–64	126,524	3,252	3,401	149	90	20,228
65–69	61,589	2,809	2,613	−196	−106	23,037
70–74	23,147	1,125	1,289	164	58	24,162
75–79	8,113	809	732	− 77	− 19	24,971
80–84	2,900	417	414	− 3	− 22	25,388
85 and over	967	137	268	131	109	25,525
	2,202,481	25,525	25,634	−445 +554		

The two extremes of the table were supplied as follows: below age 23 the ratio of actual to expected deaths was taken as 50 per cent., the value toward which the ungraduated values, when plotted, appeared to tend. For ages above 85, where the ratios exhibited intractable variations, the arbitrary assumption was made that they increased to 100 per cent. at age 95.

(*b*) *Characteristics of Provident Table.*

In the following table is shown for quinquennial ages the relation between the mortality rates of the Provident Ultimate Table and those of the American, M. A. Ultimate, $O^{M(5)}$, $O^{EM(5)}$ (*J. I. A.*, XXXVII, 142), and O^{M} Tables.

TABLE XIV.

PERCENTAGE OF PROVIDENT ULTIMATE q_x TO q_x OF OTHER TABLES.

Age.	American.	M.A. Ultimate.	$O^{M(5)}$.	$O^{EM(5)}$.	O^{M}.
20	51%	—	61%	46%	98%
25	53	91%	62	69	89
30	55	95	62	87	78
35	56	98	60	87	68
40	55	94	55	74	59
45	64	95	60	75	62
50	76	98	68	86	69
55	87	102	78	96	79
60	90	100	82	90	83
65	96	99	91		92
70	84	84	83		83
75	84	86	85		86
80	90	95	94		94
85	87	101	100		100

Spencer's twenty-one term formula, while smoothing out the irregularities, does not pretend to remove them entirely. In view of the fact that the ratio of actual to expected deaths to which the formula was applied, was based on amounts, considerable unevenness would be expected. At ages 60 to 70 a marked irregularity occurs and is probably to be explained by the large number of endowments maturing at those ages.

As a further comparison. there are given in the following table, values of the function e_x by the Provident Ultimate, American, $O^{M(5)}$ and O^{M} Tables.

TABLE XV.

e_x.

Age.	Provident Ultimate.	American.	$O^{M(5)}$.	O^{M}.
20	46.3	41.7	41.9	43.2
30	38.1	34.8	34.6	35.1
40	29.8	27.7	27.2	27.4
50	21.6	20.4	20.0	20.1
60	14.5	13.6	13.5	13.6

In 1912 a report by a committee was made to the Actuarial Society, upon the advisability of preparing a new mortality table, based upon the experience of American companies. In that report certain hypothetical rates of mortality, much lower than those of the American Table, were assumed and specimen policy values calculated; the purpose being to demonstrate with reasonable certainty that the adoption of a new table exhibiting low mortality rates would in the aggregate not reduce reserves.

The mortality rates shown by the Provident Ultimate Table bear the following ratio per cent. to those assumed by the committee.

Age.		Age.		Age.	
20	101%	50	95%	80	100%
30	92	60	106		
40	78	70	93		

Merely as a matter of interest I have calculated specimen policy values on the basis of the Provident Ultimate Table, with the following results:

TABLE XVI.

PERCENTAGE OF 3 PER CENT. RESERVES BY PROVIDENT TABLE TO THOSE BY AMERICAN TABLE.

Age.	Duration: Years.	Life.	Life, 20 Payment.	Endowment, 20 Years.	Endowment, D. or 65.
25	5	108%	98%	103%	108%
	10	107	97	102	107
	15	107	96	101	106
	20	107	95		106
	25	106			104
	30	104			103
35	5	108	102	104	107
	10	108	101	103	106
	15	106	99	101	105
	20	104	97		103
	25	102			101
45	5	105	103	104	104
	10	102	101	102	102
	15	100	99	101	101
	20	99	98		
55	5	96	99	101	101
	10	96	98	100	
	15	94	97	100	

We have in the above ratios another illustration of the general principle enunciated in 1878 by Dr. Sprague that "it is the table in which the mortality increases the more rapidly, that requires the larger policy values" (*J. I. A.*, XXI, 112). Dr. Sprague had reference to the ordinary life policy, but we observe that the remark applies also to the 20-year endowment and the endowment at 65. In the case of reserves, such as those for limited payment life, where the whole life reversion plays so large a part, the statement does not apply with equal force. In the formula,

$$ {}_{n:t}V_x = A_{x+n} - {}_tP_x a_{x+n:\overline{t-n}|}, \quad (t \lesseqgtr 20), $$

a low mortality rate is a more powerful factor than the progression of the rate from age to age.

Schedules in Workmen's Compensation.

BY

ALBERT W. WHITNEY.

A schedule is something short of a mathematical formula; the end to be attained is the same, the expression of a quantitative relationship, but usually it is not possible to make a perfect enough analysis to admit of the use of a mathematical formula. Schedules are of the greatest importance in the organization of a plan of insurance in an orderly way, particularly in such forms of insurance as workmen's compensation and fire insurance, where the problems are of a far more complex nature than in life insurance.

That a schedule may not find expression in a formula does not mean that it is not founded on mathematical reasoning. Every proper schedule is based upon a theory and this will give to the schedule a distinctive structure; that is, the elements of the problem will go together in some definite way. I believe that it is of great importance in this formative stage of workmen's compensation that the structure of the necessary schedules should be established on a proper basis and that the system should not be handicapped, as fire insurance has been, by a false logic.

In order to get these matters into the realm of discussion, I present in the present paper the outlines of two schedules that have been worked out by the California Industrial Accident Commission; both of these will at once be put into use and a third, dealing with insurance rates, it is hoped may be used before long.

I. A Schedule of Benefits.

The first and most obvious quantitative problem that one meets in the framing of a compensation law is the determination of a schedule connecting the amount of the benefit with the degree of disability. It is of the utmost importance that this schedule should fit industrial needs. If social insurance is to be the powerful instrument that we hope in doing away with poverty, relief must be applied where it is most needed; there must be no waste. And that means that such a schedule must be founded on a sociological analysis of the needs of the worker.

The relief will for the most part fall into four classes: (1) medical and surgical aid at the time of the accident, (2) payments during temporary disability, (3) payments for permanent disability, (4) payments in case of death.

In theory at least, the first and second of these are not difficult to analyze. What is needed is obvious, although there may be practical difficulties in the accomplishment. In the first case it is evident that humanity demands that the worker should have whatever relief is necessary to put him back into working condition, so far as that is possible, and it seems reasonable to suppose that such treatment will also turn out in the long run to be most expedient for the employer. In the second case what is needed is evidently such a percentage of his wages as will be sufficient to support the worker and his family pending recovery and return to work. The California schedule gives, in the first case, reasonable medical and surgical aid for a period not to exceed six months, and, in the second case, 65 per cent. of wages during the period of temporary disability, with, however, a waiting-period of two weeks to discourage malingering.

In the third and fourth cases, however, it is not so immediately obvious what the situation demands, even were the case not complicated by the compromise that must be made between what is desirable and what, at least at present, is possible.

Let us begin with the payment in case of death. This can by no means be considered under present circumstances to be full indemnity; it is only a limited insurance. In amount it will be determined by the scale of generosity of each particular state. In California the death benefit is taken to be the equivalent of three years full wages. The question now arises: how shall this be dispensed; evidently in most cases not as a lump sum. One of the faults of the liability system was that the money was so largely wasted by being paid in lump sums to persons who did not know how to use it in this form.

The death-benefit must be looked at as a family rehabilitation fund, that is, a fund that will allow the family to accommodate itself to the new condition. This change will involve time. During the period of accommodation financial independence should be assured to the family in order that the children may be properly cared for until they grow beyond the age at which they need all of the mother's attention, or until some of them may become wage-earners,

and in order that the mother may be free to plan her life so as to become a wage-earner herself. The necessary living expenses of the family have been reduced owing to the fact that the maintenance of the member who has died no longer needs to be provided; there will be some further margin of saving possible by rigid economy so that it may be assumed that the family will be able to live on 65 per cent. of the former wages. We may then agree to stretch the death-benefit over a period of 240 weeks, which would be equivalent to three years if full wages were paid. A state which can afford to be more liberal will increase this period beyond 240 weeks.

Having established the amount of the death benefit and decided how it can best be utilized, we have now to establish the proper relativity between the death benefit and the award for permanent disability. The degree of permanent disability is measured by the percentage loss of earning-power; the method of measurement is considered later in this paper.

It is assumed that a worker who has received a severe permanent injury will need for his own use 40 per cent. of his former wages; his expenses are likely to be greater than before the accident because of the need of medical attention, etc. It, therefore, follows that a man who is 60 per cent. disabled, who has, therefore, 40 per cent. of earning power left, will be able to earn only enough to support himself and, so far as the support of his family goes, he will be the same as dead. It is therefore reasonable that the family should receive the same compensation as in the case of death, namely 65 per cent. of wages for 240 weeks. We thus establish a relation between permanent disability and death. It now remains to establish a relation between the payments for different degrees of permanent disability. To do this we must first analyze the nature of the worker's needs and then determine the way in which they should be met.

In accordance with the suggestion of the Sutherland committee, it is assumed that a worker who is less than 60 per cent. disabled will be largely capable of rehabilitation, that is, it will be possible for him to regain a large part of his earning power if he can have time to learn a new trade or a new way to work at his old trade, in which he will not so severely feel the need of the lost or injured part. It is, therefore, believed that, in case the injury is less than 60 per cent., the benefit can best be applied to maintain the worker in comparative independence for a limited time during which he will be free

to make such adjustments to his new condition as may seem expedient. That such freedom will in many cases not be wisely utilized goes without saying, but it is also true that in these same cases any method of compensation would be likely largely to fail of its effect.

Under the California plan, therefore, a disability of 60 per cent. or under is compensated with 65 per cent. of wages at the rate of four weeks wages for every one per cent. of disability; a 10 per cent. disability, therefore, receives 65 per cent. of wages for 40 weeks, while a 60 per cent. disability receives 65 per cent. of wages for 240 weeks, as already explained.

In case the disability is over 60 per cent., it must be recognized that the injured worker will be for the most part incapable of any considerable rehabilitation, and that the benefits will most wisely go in the form of a life-pension that will lift the load of his support from his family. The amount of the pension should evidently be the amount by which his diminished earning-power falls short of the 40 per cent. that we have assumed necessary for his support; for example a pension of 10 per cent. of wages for a worker who is 70 per cent. disabled, and a pension of 40 per cent. of wages for a worker who is totally disabled; this is in addition to the allowance of 65 per cent. of wages for 240 weeks which is primarily for the rehabilitation of the family.

The schedule may therefore be as follows:

1. Medical aid: reasonable, for a period not to exceed six months.
2. Temporary disability: 65 per cent. loss of wages during continuance of disability, beginning two weeks from date of accident.
3. Permanent disability:
   (*a*) For a disability of 60 per cent. or less: 65 per cent. of wages for 4 weeks for every 1 per cent. of disability.
   (*b*) For a disability of over 60 per cent.: 65 per cent. of wages for 240 weeks and thereafter a life pension of one per cent. of wages for every one per cent. of disability in excess of 60 per cent.
4. Death: 65 per cent. of wages for 240 weeks (to go into a state fund for accident prevention in case there are no dependents).

This is essentially the California schedule, except for one or two slight changes made on the ground of practical expediency. In theory, to be consistent with the rest of the plan, the life pensions should begin at once, but in the California schedule they begin at the end of 240 weeks. In practice this has some advantages; for instance, it puts everything, temporary disability, permanent disability, and death, on a 65 per cent. basis from the start, so that if temporary disability runs into permanent disability or if either temporary disability or permanent disability results in death, the adjustment is perfectly simple.

Let us now see whether the awards for permanent disability are reasonably consistent with each other. To do this we must compare the lump sum equivalents of the various awards. It would be instructive to work this out for a variety of ages; in that case we should find that for young ages the awards for severe permanent disability were, relatively to those for slight permanent disability, more important than in the case of old ages, and this is doubtless as it should be. We shall obtain an average result, however, and a result which may be used in estimating the cost to the employer of permanent disability as compared with that of temporary disability and death, by taking the age for which the value of a life annuity is the same as the average value of a life-annuity for all those permanently disabled. This age by the California accident experience, American Experience Table, 3½ per cent., is about 39, and it happens this is exactly the age that had already been assumed based upon an actuarial analysis of census returns.

For age 39 the lump sum equivalents of the benefits for the various degrees of permanent disability by the American Experience Table, 3½ per cent. interest, are as follows; for three different schedules, (1) the schedule proposed in this paper; (2) the present California schedule; this was founded upon the Wisconsin law and is to be superseded January first by the new schedule; the old schedule pays 65 per cent. of the loss of wages, limited in amount to 3 years full wages and in time to 15 years; (3) for comparison, a schedule paying half the loss of wages:

TABLE I.

Percentage of Permanent Disability.	Lump Sum Equivalent of given Compensation in Week's Wages.		
	1. Proposed Schedule.	2. Present California Schedule.	3. Schedule Paying Half Loss of Wages.
10	25	37	45
20	50	74	89
30	74	110	134
40	97	122	179
50	119	128	224
60	141	132	268
70	209	135	313
80	276	138	358
90	344	139	402
100	412	141	447

If these values are plotted it will be noticed that (3) gives a straight line, that (1) gives a broken line bending up, and that (2) gives a curved line bending down. Abstract justice to the worker probably demands a distribution similar to (3). It may be questioned, however, whether, at present, compensation can be looked at in such a light and whether it must not, perforce, rather be looked at largely as an instrument to be used by society for averting poverty. If so the criterion of efficiency will doubtless demand that those that are slightly disabled shall give way before the superior need of those that are severely disabled. This principle has received the endorsement of the workers* themselves.

The proposed schedule carries out this principle by paying relatively far more for the severe disabilities than for the slight disabilities. It will be noticed that schedule (2) does exactly the wrong thing, and compensates the slight injuries far more liberally relatively than the severe ones. In fact, under the present California schedule, the total amount receivable for any disability

* For example I quote from a published report of the Pennsylvania Federation of Labor:

"There was a general consensus of opinion among the labor delegates at the conference that they would prefer to be assured of a pension for life in case they were totally incapacitated for life even if it were necessary to suffer a reduction in the compensation for minor injuries. It is suggested that within the limits spoken of above (the maximum limit of cost to the employer fixed by the rate of insurance) working-men ought to be free to agree among themselves that they will waive their claim for compensation if a finger is cut off so that they may get a greater compensation in case an arm is amputated. What can be saved in minor accidents can be used, in part, to increase the compensation for those completely crippled for life."

over 30 per cent. is the same, the only difference being the period over which it is paid.

The upper and lower limits are still to be fixed. Far the simplest and most consistent method seems to be to fix an upper and lower limit of wages. In California these limits are $333\frac{1}{3}$ and $1,666\frac{2}{3}$, that is, a person getting less than $333\frac{1}{3}$ per year is compensated on the basis of $333\frac{1}{3}$ and a person who gets more than $1,666\frac{2}{3}$ is compensated on the basis of $1,666\frac{2}{3}$.

A schedule of this kind has several virtues besides those that have appeared in the analysis. It has, for one thing, a high degree of flexibility; by that I mean that without a departure from the essential plan it can be modified to suit local conditions. Suppose, for instance, that it is desirable to effect a reduction in the cost of the schedule. This can be done in several ways, most obviously by changing the percentage of wages allowed from 65 to 60 or to some other figure. This will make a uniform cut in the compensation all along the line except in the case of the life pensions, and these can be similarly modified if desirable. An alternative method would be to shorten in some uniform way the periods during which the payments for death and permanent disability are made. This would leave the payments for temporary disability and the life pensions unchanged. A still further method of control would be to decrease the upper limit of wages. This would have the effect of making the saving come from the more highly paid. We should then have an interesting sociological question to decide, whether the consequent lowering of the standard of life for such families might not possibly do more harm than a less severe cut that would affect all alike.

It is not claimed that the details of this schedule are necessarily the best possible, nor that there may not be better analyses of the worker's needs; it is put forward only as being perhaps one of the first attempts to coordinate the various benefits on the basis of a definite theory.

It is not a part of the present California plan to take account of the degree of dependency; that is a step that is further in the direction of making compensation poor relief than is at present contemplated.

The payment of the death benefit to the state in case there are no dependents is to prevent adverse selection by the employer against married men.

Some interesting actuarial results were got as to the cost of the new California schedule, and particularly with regard to the relative cost of the three typical schedules of permanent disability that have here been given and in comparison with the schedules of other states. For this purpose an accident experience was assumed based upon a combination of German, Italian, Austrian and Russian experience. Because it has been so often urged that European experience has no value when applied to this country, it may be interesting to set down the fact that the experience assumed was in very close agreement with the California experience upon some 10,000 accidents that later became available and gave substantially the same results as to cost. I will not attempt to present the details in the present paper.

II. A Schedule of Disability.

The schedule of benefits is based upon a rating of disability, the rating being expressed as a percentage loss of earning-power. It becomes then a very real and pressing problem to be able under varying circumstances to assign ratings to various physical disabilities. The varying circumstances that we are in practice warranted in recognizing are age and occupation, that is, in other words, we must construct a schedule in which the percentage loss of earning power will be expressible as a function of the physical injury, the age, and the occupation. In most schedules of benefits the loss of earning power is made to depend upon the physical injury alone. This is altogether too crude. For instance, the loss of the index finger is a much more serious matter to a printer than it is to a stevedore, and the seriousness of the accident is also greatly increased if the worker is so old as not to be able easily to learn a new trade, or, in the case of the printer, to learn to set type with a new combination of fingers.

Evidently the effect of advanced age upon the problem is principally in its lessening of the power of accommodation. In order to get the two extremes we will assume first that a boy of 15 has perfect accommodation, that is, that being just about to learn a trade he can with entire freedom turn to another. On the other hand we will assume that a man of 75 has no accommodation, that he cannot turn to a new trade. So much at present for age.

The effect of occupation upon the problem evidently lies in the

degree of specialization that it produces in the different members. The problem of rating this will presently be discussed, but first we must define the standard man. We will take as a standard man a man who has no accommodation, not necessarily to be thought of as a man of 75, but a man who does not turn to a new occupation; furthermore, our standard man shall have no specialization of his members, that is, he shall be an average all-round man.

Let us now for such a man evaluate the effect upon the earning power of various physical injuries. This is one of the most difficult parts of the problem. It is now being worked out by the California Industrial Accident Commission in consultation with physicians, employers, employees, the accident boards of other states, and through reports of the practice in foreign countries that are being specially obtained through the consular service.

Suppose, however, that the standard ratings have been established. Let us now vary the age, still having in mind however the average, unspecialized man. The problem here is to define the divergence from the various standard ratings for various ages. At the two extremes we have equality: What is a zero disability at age 75 will also be a zero disability for all other ages. Furthermore, what is a total disability at age 15 will also be a total disability for all other ages. But intermediate disability values will evidently be less the younger the age. How much less is a matter for conjecture or experiment. In the California schedule the following assumption has been made: that what is a 20 per cent. disability for a man of 75 will be only half of that for a boy of 15, that is, that a boy, by virtue of his power of accommodation, can recover in this case half of the earning power he has lost. This assumption, with the values as determined at the extremes, is sufficient to define a quadratic equation—the simplest function possible—which will give all other values at age 15. The equation is $y = (5x^2 + 3x)/8$, where x is the fractional value of the loss of earning power for a physical disability at age 75 and y is the corresponding value for age 15. The values for intermediate ages are got by a linear* intrapolation. A few values of the resulting table are as follows:

* Another law might be used if it seemed best.

TABLE II.

Standard table giving comparative percentage Degrees of Disability for an unspecialized Worker.

Age.....	15	27	39	51	63	75
	Per Cent. of Disability.					
Line I.....	10%	12%	14%	16%	18%	20%
Line II.....	25	28	31	34	37	40
Line III.....	45	48	51	54	57	60
Line IV.....	70	72	74	76	78	80
Line V.....	100	100	100	100	100	100

The table says, for instance, that what is a 60 per cent. disability for a man of 75 is only a 45 per cent. disability for a boy of 15 with perfect accommodation, and a 51 per cent. disability for a man of 39. Instead of the given assumption—namely that a boy of 15 can recover half his earning power in the case of what would be a 20 per cent. disability if there were no accommodation—any other assumption may equally well be made, and the other values may be determined by any other law. However, nothing has appeared to indicate that the table as here given is seriously out.

We have now taken account of age so far as it affects accommodation. There is in addition an effect of age upon the power of recovery. This so far as it does not depend upon the nature of the particular disability may be considered to be incorporated in the preceding table. The effect of age upon the power of recovery so far as it is peculiar to specific injuries can hardly be taken account of in a systematic way without making the schedule too complex.

We now take up the effect of occupation upon disability. Let us suppose for an example that the loss of the first two fingers of the major hand is a 20 per cent. disability for the standard man, that is, for an unspecialized man of 75. But these two fingers are more valuable to a typesetter than to the average man, and we may suppose that for a typesetter without accommodation this physical injury should count as a 60 per cent. disability instead of 20; that is, we will give the loss of the first two fingers a + 40 rating for a typesetter. For a typesetter of 15, however, the loss of his two fingers is no more serious than for the unspecialized man, for in reality in his case the specialization has not yet begun. For intermediate ages we may, as before, suppose a linear intrapolation, or something more complicated if any reason should appear. We may

incorporate the results of an increased rating of 40 upon the various standard ratings in the following table:

TABLE III (+ 40).

Age.....	15	27	39	51	63	75
	Per Cent. of Disability.					
Line I.....	10%	20%	30%	40%	50%	60%
Line II.....	25	36	47	58	69	80
Line III.....	45	56	67	78	89	100

Lines I, II, III indicate standard ratings of respectively 20, 40 and 60, as before. Lines IV and V would evidently serve no purpose in this table. Similarly + 10, + 20, + 30, + 50, and other tables can be made in a similar way.

In the case of certain occupations a particular disability may have a minus rating, for instance, the loss of a leg to a bookkeeper has probably less effect upon his earning power than in the case of the average man. A table of minus ratings may therefore be made in a similar way to the table of plus ratings; for example:

TABLE IV (— 10).

Age.....	15	27	39	51	63	75
	Per Cent. of Disability.					
Line I.....	10%	10%	10%	10%	10%	10%
Line II.....	25	26	27	28	29	30
Line III.....	45	46	47	48	49	50
Line IV.....	70	70	70	70	70	70
Line V.....	100	98	96	94	92	90

It will be noticed that the last line illustrates the fact that a rating might under these circumstances be less the more advanced the age. For example, in the case of a bookkeeper it might well be that the loss of a leg might be a more severe handicap to a young man than to an older man who is established in his profession. The older man would hardly be thrown out of his place because of the disability, while it might be impossible for the young man to make a start. However, as a practical matter, I do not believe it would be wise to recognize such cases, and thus line V in the last table should read 100 throughout.

In applying these tables it will be necessary to give a standard rating to every physical disability and to qualify it by a plus or minus rating for every occupation. As a matter of practice it is expected that the occupations will fall into a comparatively small number of classes. A fragment of such a table might appear as follows:

TABLE V.

Physical Disability.	Standard Rating.	Occupation.			
		Printer.	Stevedore.	Structural Iron Worker.	Locomotive Engineer.
Loss of index finger.....	Line I	+40	0	0	0
Loss of leg.....	Line III	− 5	0	0	0
Loss of major arm.....	Line IV	0	0	0	0
Partial loss of balancing power.....	Line I	0	0	+40	+20
Slight tendency to epilepsy (caused by accident).....	Line II	0	0	+40	+60

Zero of course means that a rating is standard.

The schedule is to be used in the following way; take an example: A tendency to epilepsy produced by an accident is here listed as a 40 per cent. disability for the standard man, and with regard to this particular disability a printer and stevedore are classed as standard, but a structural iron-worker is given a + 40 rating and a locomotive engineer a + 60 rating, making in the latter case total disability for age 75. If then a stevedore of 39 has suffered this kind of accident, we are directed to use the rating for age 39 of line II of Table II, namely 31, while if the same accident has happened to a structural iron-worker of 51, we are directed to use the rating for age 51 of line II of Table III or 58.

It might also be noted that a table of this kind can be used for rating the disability produced by a combination of injuries to several parts of the body. The result is not to be obtained by simple addition; on the one hand in such cases as that of the eyes and the arms where one member can be trained to do the work of the other, the loss of both members would be more than the sum of the disabilities produced by the loss of each. On the other hand, where the injuries are to parts of the body that have independent functions, the addition of the two figures would give too high a rating. For instance, suppose the rating for a left hand were 40 and for a leg

60, the rating for the loss of both should hardly count as a total disability as it would if simple addition were used. We may instead use the following analysis: a man who has lost a leg has only 40 per cent. of earning power left. The loss of the left hand will take away 40 per cent. of his remaining earning power or 40 per cent. of 40 per cent. or 16 per cent. The loss of both hand and leg will thus count as a 60 + 16 or a 76 per cent. disability.

For organs having the same function it will be necessary to treat each separate combination; for example, both eyes, both arms, etc.; but as there are only a few of such combinations these could easily be all set down in the schedule of permanent disabilities.

A great deal of careful work will be necessary in making a schedule of this kind, but when it is once done it is believed that we shall have an instrument which will be easy to handle and which will do substantial justice in the great majority of cases. It must be remembered in this connection that, in spite of the fact that the problem is a difficult one, it *must* be solved, that is, judgments with regard to the degree of disability must be made, and anything that can be done to systematize the process will be a clear gain; on the other hand, we must be held in check lest we produce a system which, while it may be more accurate, will be too complex to be workable.

It need hardly be said that the figures here used are mainly in the way of illustration of the method and are not to be taken as final.

The Influence of Medical Re-examination on Insured Lives: Health Conservation.

BY

CHARLES W. JACKSON.

The idea of conserving the health and efficiency of policyholders in a life insurance company by an educational campaign and medical re-examination originated with Dr. Eugene L. Fisk, then Medical Director of the Provident Savings Life Assurance Society, in 1907, and with the co-operation of Mr. E. E. Rittenhouse, President of the above Society, was put into active operation in the year 1909.

It has been continued by the Postal Life Insurance Company, of which company Dr. Fisk is the Medical Director. Bulletins containing information on sanitation, hygiene, prevention of avoidable disease, and other medical and scientific information, written in a form which the layman can understand, are issued periodically and sent to all policyholders. The latter have also the privilege of medical examination once a year, if applied for within two months of the anniversary of the policy and payment of premium then due.

When the applicant requests this medical examination, he is referred to an examining physician and asked for the name of his own family physician, and permission is requested to notify such physician in case impairment is found. The result of the medical examination is sent by the examining physician to the home office, and a specimen of urine forwarded by mail in a celluloid container, containing a preservative tablet. If no impairment is found, the applicant is notified of the fact. If an impairment is discovered, he is advised to consult his family physician, who is notified of the impairment. The applicant being thus warned of serious trouble, consults his physician and is immediately advised of the necessary steps to take to ward off threatening disease, if he is not already aware of it.

By this means, incipient diseases, such as diabetes, Bright's disease, heart trouble, and diseases of the lungs are discovered and treated in their early stages, and often times cured. In nearly half the cases, the examined had no idea that their health was impaired,

and, in the ordinary course of events, would not have consulted a physician until the disease had made considerable headway, perhaps too great to permit of cure.

Many letters have been received from policyholders who have taken advantage of the privilege, and from their wives, expressing gratitude for the prevention of serious illness, if not untimely death.

Upon receipt of an application for medical re-examination at the home office, a card is filled out containing the name and address of the applicant, the number, amount, and kind of policy, the date of birth, date of issue, and later the date and result of re-examination.

The accompanying card is a copy of one made for one of the risks who was re-examined and died within two months of examination:

No. *43636* Age of Entry *23* Loss Number

Name Born *12/13/68* State or Country

AMOUNT		KIND	DATE OF ENTRY			DATE OF EXIT			MODE OF EXIT	CAUSE OF DEATH
1	*000*	*Y. R. T.*	*1*	*25*	*1892*	*3*	*22*	*11*	*D.*	*Tuberculosis*

0 1 2 3 4 5 6 7 8 9 DURATION

PREMIUM

9 × *5.11.143* *115*

1 × *7-4* *8a*

a. 61 *70* *B. 31*

a. 49

Exam. H. B. applied for 1/7/1911
Advised 2/1/11

Occupation *Banker* Residence

The code shows that he was 5 feet 11 inches and weighed 143 pounds at date of issue, but only 115 pounds at date of re-examination; that he had tuberculosis and albuminuria at date of re-examination; that one parent (probably father) died at age 61 of paralysis; that the other parent died at age 49 of cancer, and that a brother died aged 31 of consumption; that application for medical re-examination under the Health Bureau was received on the seventh of January, 1911, and that the insured was advised of the result on the first day of February of that year. This particular case is included in the $103,521.00 actual mortality experienced among the impaired risks.

An analysis of the experience covering four years has been made. The results are shown in the following table:

	Exposed to Risk.	Expected Mortality Am. Exp.	Actual Mortality.	Percentage of Actual to Expected.
Impaired	$4,797,980	$110,146.93	$103,521	93.98%
Unimpaired	5,196,512	98,463.80	29,834	30.3
Total	9,994,492	208,610.73	133,355	63.93
Average Attained Age, date of Analysis:			Impaired 51.75	Unimpaired 45.81

I find upon inspecting the deaths among those found impaired that:

3 insured for $3,000 died within 2 months of examination.
6 insured for $18,834 died within 4 months of examination.
4 insured for $8,000 died within 6 months of examination.
4 insured for $15,500 died within 12 months of examination.

If we exclude those who died within six months of the examination of serious illness disclosed by the examination, and who thus were beyond the influence of preventive medicine, we find that the actual mortality among the rest of the impaired was $73,687, or about 66.9 per cent. of American Experience mortality, the mortality in the whole group was $103,521, or about 50 per cent. of the expected. The average age at death of those dying within six months was 60.7 years.

To what extent this result is due to a change in living induced by the educational campaign carried on by means of the bulletins, the knowledge of impairment and measures taken in consulting with the doctors, or to the temperaments of those who avail themselves of the privilege, it is hard to determine, but it seems only reasonable to conclude that it is to a very large extent due to the former. It must also be borne in mind that the average exposure was only two years; yet in this connection it is worthy of note that the present analysis shows substantially the same results as one undertaken more than a year ago.

The practical question will arise as to the cost of the health bureau. Accounts are carefully kept and the bureau is charged with the expense of printing and circulating the bulletins and with medical examiners' fees and incidental expenses. During the four years, the expense of medical re-examination (an influence which

can be measured), was $7,393.10, consisting of medical examiners' fees of $5,732; the cost of containers, postage and incidentals, $1,661.10.

The problem as to the degree of mortality saving among the risks medically re-examined is new in actuarial science and in the science of preventive medicine. It is not fair to assume an expected mortality in accordance with the experience on ordinary substandard lives, or even upon rejected lives, for the reason that many of the impaired lives were far beyond the age of insurability and so badly impaired that a number of them died within a few months after examination, as stated above. It would be possible, of course, to group some of these lives under ordinary substandard impairments and approximate the expected mortality, but it is extremely difficult, if not impossible, to approximate the expected mortality of those far advanced in disease.

Such a grouping and such an analysis, which is an exceedingly complex affair, is now being made by Dr. Fisk. In the interval, he has felt justified in assuming an expected mortality on the entire impaired group of 200 per cent., realizing that this is a guess, yet, in his judgment, a guess well within the mark.

A mortality of 175 per cent. has been found among a group of rejected lives by a large life insurance company.

A mortality of 200 per cent. has been experienced in a group of substandard business insured under lien policies in another company.

It should also be borne in mind that no trivial impairments were recorded on the cards.

The saving arising from these re-examinations, according to various possible estimates of the expected mortality, would be as follows:

Expected Mortality Assumed.		Actual Mortality.	Gross Savings.	Net Savings.
125%	$137,683.66	$105,321	$ 34,162.66	$ 26,769.56
150%	165,220.39	105,321	61,699.39	54,306.29
175%	192,757.12	105,321	89,236.12	81,843.02
200%	220,293.86	105,321	116,772.86	109,379.76

The saving in the group of unimpaired lives, the mortality among which was 30.3 per cent. of American Experience mortality, arises from the improvement in living habits. Many of the risks were not

impaired in the strict sense of the word, and were grouped as unimpaired, yet actually were not living strictly hygienic lives; for example many of these risks were advised to change their habits in regard to the use of food, tobacco, stimulants, also with respect to exercise, sleep, work, etc. These risks were chiefly drawn from a group that had been long insured, of a fairly high average age, and exhibiting for these reasons a high general mortality, nevertheless, the mortality among the re-examined lives exhibits them in a super-standard group.

The educational campaign carried on by means of the health bulletins also gives a considerable amount of favorable publicity to the company, and much new business is traced directly to this source.

The analysis of this experience throws an interesting light on the question of lapse. It has been frequently maintained, and even recently at the meetings of the Society, that the healthy alone lapse, and that those whose health is impaired do not allow their policies to lapse, but take advantage of the loan value, or their friends pay their premiums. In any case, they remain on the books of the company. This has not proved to be the case. During the four-year period, $5,185,054 of the impaired were exposed to risk of lapse, while $404,000 lapsed, a percentage of 7.79; of the unimpaired, $5,373,740 were exposed, while $375,500 lapsed, a percentage of 6.99. It will thus be observed that a smaller percentage of the unimpaired, who enjoyed a very low mortality indeed, lapsed, than of the impaired. This proves conclusively that in the case of the risks under consideration, other factors than questions of health had a powerful influence on the lapse-rate, for, in this special experience, all of the impaired who lapsed were aware of the fact that they were impaired risks.

Legal Notes.

BY

WENDELL M. STRONG.

Designation of Beneficiary:—(Mutual Benefit Life Insurance Co. *vs.* Cummings, Supreme Court of Oregon, 133 Pac. Rep. 1169.) The insured had taken out a policy naming as beneficiary Evelyn M. Cummings, "his wife." The insured was really separated from his wife and was living with a woman known as Evelyn M. Cummings whom he introduced as his wife. His wife, Sophia J. Cummings, claimed the proceeds of the policy. The question was to which the policy was payable. On the one hand, insured's wife was Sophia J. Cummings. On the other hand, he was living with Evelyn M. whom he introduced to friends and acquaintances as his wife, and the name, Evelyn M. Cummings, was in the policy. Furthermore, he had been for years estranged from Sophia J. Cummings, and she did not even know of the existence of the policy until after the death of the insured. The court held that there was no doubt that the beneficiary was the woman who passed under the name Evelyn M. Cummings, and that the immoral relation did not vitiate the contract.

Recovery of Statutory Penalty for Refusal to Pay Policy:—(Georgia Life Insurance Co. *vs.* McCranie, Court of Appeals of Georgia, 78 S. E. Rep. 1115.) Insured's death was caused by taking carbolic acid. Payment of the policy was refused on the ground of suicide. The verdict of the jury was against the company, and this verdict also awarded the statutory penalty because the refusal to pay was in "bad faith."

The question which interests us in the case is that of the statutory penalty. The company had offered the testimony of its general counsel regarding the investigation it had him make, what he had learned, and his advice to the company. This testimony had been excluded. On appeal the court held that this testimony should have been admitted. It also held that there had been strong evidence presented tending to show that death was by suicide, that, under

such circumstances, the company could not be regarded as acting in bad faith in refusing to pay, and that, consequently, the statutory penalty should not be inflicted. In the opinion it quoted with approval from a Missouri decision as follows: "It must be borne in mind that the statutory punishment is not inflicted merely for the reason that it turns out at the trial there was, in reality, no reason for the delay. The question is, How did matters appear before the trial, as judged by a prudent and reasonable man seeking to find out the facts about an occurrence which it was his duty to investigate?"

The danger of incurring a statutory penalty for refusal to pay on the ground that such refusal was in bad faith, even where the company had strong grounds for believing it was not liable, has been considerable, and the present decision with the strong opinion accompanying it should tend to lessen this and to render the companies more free to contest claims which they have reason to believe unfounded.

Construction of Ambiguous Wording in Contract:—(Hay *vs.* Meridian Life and Trust Co., Appellate Court of Indiana, 101 N. E. Rep. 651.) The danger of a clause in the contract which is not specific is well illustrated in this case. The court based its decision upon the well established rule that a clause of doubtful meaning must be construed against the company, since the wording of the contract is determined by the company. It would seem, however, that the application of the rule in this case was extreme.

A policy was dated back seven years, and payment of the premiums for the seven years was made by means of a loan on the policy. Two premiums were paid in cash and then the policy lapsed. The insured died in the following year. It was claimed that the insurance was in force under the extended insurance clause. The company denied liability. The policy stated: "Any indebtedness on account of this policy . . . will reduce the values specified herein, in the proportion that such indebtedness bears to the amount upon which the above guarantees are based; the extended insurance being reduced proportionately in amount of insurance in force, not in the term of extension." The decision of the court against the company turned upon the wording "to the amount upon which the above guarantees are based," the court holding that the word "amount" might well apply to the face of the policy and that an insured might easily so interpret it.

This interpretation seems unreasonable. The indebtedness upon the policy exceeded the cash value, and the extended insurance was an alternative to the cash value. The policyholder would not expect to have extended insurance when he knew that he had already in the amount of the loan received more than the full cash value stated in the policy.

What Constitutes Acceptance of Application:—(Kennedy *vs.* Mutual Benefit Life Insurance Co., United States District Court, District of Montana, 205 Fed. Rep. 677.) The applicant paid the premium and was given a receipt which stated:

"This receipt will be binding on the company from the date of medical examination, provided the application for the insurance is approved and policy issued by the company, as applied for."

The application was approved at the home office, but in the policy department it was noted that a microscopical examination was required by rule and that it was lacking. The application was referred back and the approval canceled. A letter was sent to the agent in Butte stating that the microscopical examination was desired. Four days after the sending of this letter the applicant was killed. The court, hearing the case without a jury, held that the insurance was not in force, since the policy had not been issued, stating that, until the policy was issued or the applicant notified of the acceptance of his application, the acceptance by the company was not complete.

Misrepresentation: Interpretation of Missouri Statute:—(Conner *vs.* Life and Annuity Association, Kansas City Court of Appeals, 157 S. W. Rep. 814.) This case is an illustration of the extreme interpretation of the Missouri statute providing that no misrepresentation in securing a policy shall be deemed material or render the policy void unless the matter misrepresented has contributed to the contingency making the policy payable, and that the question of such contribution shall be for the jury.

The policy was dated December 6, delivered December 7, and the insured died December 29. Insured stated in his application that he had not been under treatment at any asylum, cure, or sanitarium, and that his father was forty-two years old and in good health, and that he had a sister twenty-four years old in good health. It was admitted at the trial that, in the spring before the policy was issued, he had been three months in an institution for treatment of sufferers

from tuberculosis, that his father had been dead thirteen years, and that his only sister had been dead five or six years. There was also evidence that insured died of tuberculosis, and had it at the time he signed the application, which evidence the court mentions as "strong but by no means conclusive," a description which, in view of the synopsis of the evidence given in the opinion, seems very conservative.

The jury, however, in the trial court gave a verdict for the plaintiff, and, on appeal, this was affirmed. In the course of the affirming opinion, and in regard to the statements admittedly false, the court said that, while in certain previous cases it had held that "insurance obtained by any fraud remediable at law or in equity may be avoided on a proper showing of the fraud, and the statute on misrepresentations does not preclude the defense of fraud in obtaining a policy," it was now bound by a later decision of the Supreme Court which overruled these cases, and which interpreted the statute as covering fraudulent misrepresentations, with the result that "during the life of the insured, but not afterwards, the company may sue to set aside a policy on the ground that it was procured by fraud."

The logical result of this doctrine may, not unfairly, be said to be, that the worse the fraud the less the chance of the company to defend because of it. For, when a person close to death succeeds in procuring a policy by fraud, the chance of defending on the ground of fraud is practically taken away: since it is extremely improbable that in such cases the company will be able to discover the fraud and sue to set aside the policy before death occurs; and after death occurs the proof of fraud is not sufficient.

Extended Insurance; Effect of Part Payment of Premium Due:—(Taylor *vs.* New York Life Insurance Co., Court of Appeals of New York, 102 N. E. Rep. 524.) Three annual premiums on the policy in question were paid; when the fourth premium came due, it was paid partly in cash and partly by a six-months' note; when this note came due, part of it was paid in cash and a three-months' note given for the remainder, which note was not paid when due. The insured died one day after the end of the period for which extended insurance would have been granted on default in payment of the fourth premium. The question was whether the cash payments toward the fourth premium extended the period.

The notes contained the provision:

"This note is given in part payment of the premium due December 20, 1898, on the above policy, with the understanding that all claims to further insurance and all benefits whatever shall become immediately void and be forfeited to the New York Life Insurance Company if this note is not paid at maturity, except as otherwise provided in the policy itself."

The court held that the cash payment, which amounted to one half of the annual premium, carried the period of extended insurance at least one day beyond the date when it would have expired had default occurred at the end of the third year. It, however, left in doubt the precise method of determining what the period of extended insurance would have been, as this did not need to be settled. It referred to the New York statute and the finding of the trial court that the 4½ per cent. reserve at the date when the second note matured, less the indebtedness represented by such note, would have carried the insurance beyond the date of death. It also stated, apparently as an alternative proposition, that it might be assumed that the reserve on the date to which the cash payment would have paid the premium would, under the New York statute, have carried the insurance beyond the date of death of the insured.

Several points are worthy of notice in this decision. First, that the wording of the note did not show that a failure to pay it would work an unconditional forfeiture of the cash paid upon the fourth year's premium. Second, that the cash paid constituted a part of the fourth year's premium. Third, it is left unsettled whether the calculation of the extended insurance should be on the basis of default at the date to which the cash payments were sufficient to pay the premium, or on the basis of taking the reserve at date of maturity of the second note, subtracting from this the indebtedness on such note, and using the balance to purchase extended insurance in accordance with the New York statute.

Extended Insurance; Date of Termination:—(Cady *vs.* Travelers Ins. Co., Supreme Court of Nebraska, 142 N. W. Rep. 1007.) After default in payment of premium, correspondence took place between the insured and the company, the company urging that the premium be paid. The question was whether the term of extended insurance began at date of default in payment of premium, or whether the correspondence, and the attempt by the company to have the policy continued by payment of premium, fixed a later date as the beginning of the extended insurance.

The court, in a well-considered opinion, held that the extended term dated from default in payment of premium, and that the distinct provisions of the contract to this effect were not waived by correspondence urging the insured to continue payment of premium.

The question involved is of considerable importance, as any other decision would have resulted inequitably; for a postponement of the date at which the extended insurance begins to run would give a longer period of extended insurance than that contracted for, in other words, would give a larger surrender value than that provided for by the contract.

AGREEMENT TO ACCEPT POLICY IF ISSUED:—(Citizens National Life Ins. Co. *vs.* Murphy, Court of Appeals of Kentucky, 156 S. W. Rep. 1069.) The application contained an ordinary form of agreement to accept the policy if issued. The applicant had executed a note for the amount of the premium, but refused to accept the policy when issued. Some time later the policy was mailed to him, and he retained it for some six months, then returned it. The application contained the provision that the policy was valid only if delivered and paid for during the good health of the applicant.

It was held that the application was a proposition and imposed no obligation on the applicant unless accepted by the company. In view of the condition about delivery while in good health, the approval of the application and issuing of the policy was not unconditional acceptance, and hence it was still open to the applicant to withdraw his proposition when the policy was offered to him. He did this, consequently the consideration for the note failed, as there was no insurance. The fact of the mailing of the policy after the applicant had refused to accept it was of no effect.

CANCELLATION OF POLICY:—(Palmer *vs.* Mutual Life Ins. Co., Supreme Court of Minnesota, 141 N. W. Rep. 518.) A paid-up policy having no guaranteed surrender value was assigned as security for a loan. The loan agreement provided that in case of default in payment of the loan the company could cancel such policy and apply the customary value (which was stated in figures in the agreement) allowed for such policies, to the payment of the loan with interest, the balance of such value, if any, to be paid in cash. When default occurred, the company cancelled the policy and tendered the insured the balance of the value. The insured objected to the can-

cellation and refused to accept the check. Somewhat more than three years later the insured died, and suit was brought to recover the amount of the policy less the amount of the indebtedness. The court held that recovery could he had on the following grounds: (1) that the actual value of the policy was greater than the surrender value allowed by the company; (2) that the transaction was a pledge; (3) that being a pledge, an agreement for cancellation for less than the full value was not effective; (4) that the cancellation being ineffective the policy continued in force.

Who is Entitled to Proceeds of Policy when One Beneficiary of a Class Dies Before the Insured? (Hartung et al. *vs.* Northwestern Mutual Life Ins. Co., St. Louis Court of Appeals, 156 S. D. Rep. 980.) The policy named as beneficiary insured's wife and his children by her. It contained the clause "And the said company doth hereby promise and agree to pay the said sum assured, at its office, to the said beneficiary or their executors, administrators or assigns, in sixty days after due notice and proof of death of the said person whose life is hereby assured. . . . In case of the death of the said beneficiary, before the death of the person whose life is hereby assured, the amount of the assurance shall be paid at maturity to the heirs or assigns of the said person whose life is assured." The wife and four children survived the insured, one child having predeceased him. The question was, what became of such child's share of the policy. The court held that the beneficiary named was a class, and that there could be no divestiture unless all of that class had died prior to the insured, and, hence, that the share of the deceased child went to that child's estate. In support of this, emphasis is laid on the company's promise to pay to the "*beneficiary* or *their* executors, administrators or assigns" and the words in the divestiture clause "in case of the death of said *beneficiary.*" It was said that in these clauses "beneficiary" is in the singular, so that the death of the beneficiary could be said to have occurred only when all the class named had died; that the words "*their* executors etc." indicated also that the policy contemplated the possible death of one or more of the class prior to that of the insured and made provision as to the share of that one, that it should go to his or her executors, administrators or assigns.

It would seem that in the interpretation of the word "their" as implying a provision for the share of one who died prior to the

insured, the contingency which this word was undoubtedly intended to cover was overlooked, namely, that one or more of the class might die after the death of the insured but before payment of the policy had been made.

TAXATION; "DOING BUSINESS" WITHIN A STATE:—(Commonwealth *vs.* Provident Savings Life Assurance Society, Court of Appeals of Kentucky, 159 S. W. Rep. 698.) The question was that of taxation of life insurance premiums. The company had withdrawn from Kentucky and ceased to do business there. The statute provided that every company "doing business" in the state should pay a tax upon all premiums received "in this state or out of this state on business done in this state." The court held, as regards the words "doing business":

"Having succeeded in taking risks in the state through a number of years, it cannot be said to cease doing business therein when it ceases to obtain or ask for new risks, or to issue new policies, while at the same time its old policies continue in force, and the premiums thereon are continuously paid by the policyholders to an agent residing in another state, and who was once the agent in the state where the policyholders resided. This action on the part of the company constitutes doing business within the state, so far as is necessary, within the meaning of the law upon this subject."

This should be compared with the Wisconsin decision upon a similar point under a statute reading somewhat differently (*T. A. S. A.*, XI, 507) and with the decision of the Supreme Court of the United States (*T. A. S. A.*, XII, 95).

Furthermore, the company did not collect any premiums in the State of Kentucky, the premiums on its policies in force in that state being received in New York. The court held regarding this:

"Section 4226 of the Kentucky Statutes was a part of the law of this commonwealth when the appellee entered the state and obtained therefrom its license to do business as an insurance company, and this section became a part of the contract between the insurance company and the state, and the appellee impliedly agreed that all premiums receipted for on the face of the policy for original insurance, and all renewal premiums received in cash or otherwise in this state, or out of this state, on business done in this state on all policies which have been written in, or on, the lives of residents of this state, or out of this state, on business done in this state, shall be subject to taxation. That is, it agreed to pay into the state treasury a tax $2 upon each $100 of said premiums as ascertained."

The soundness of the decision and the enforceability of the judgment in another state are both doubtful.

MATERIALITY OF REPRESENTATION:—(Rigby *vs.* Metropolitan Life Ins. Co., Supreme Court of Pa., 87 Atl. Rep. 428.) The insured in his application stated he had suffered from dyspepsia and gave the name of the attending physician. He gave negative answers to the questions "Have you consulted any other physician?" "Have you had any other medical attendant or have you been prescribed for by any other physician than the above named?"

Insured had, shortly before the application was made, consulted another physician six times, and it appeared that this physician did not regard his disorder as a trivial one. The court held that the incorrect answers were material as a matter of law, saying:

"The questions alleged to have been incorrectly answered were material to the risk, and their materiality was so manifest that it was the duty of the court to direct the jury to find for the defendant if the testimony of his witness was believed, unless there was something to take the case out of the clearly established rule. We do not find that there was. The illness for which the insured sought the aid of Dr. Harvey was not a trivial one, and so unimportant as to have no relation to his general health."

REINSTATEMENT; NOTICE OF ACCEPTANCE:—(M'Cormick *vs.* N. Y. Life Ins. Co., Supreme Court of New York, Appellate Division 141, N. Y. Supp. 993.) The policy in question lapsed, and insured made application for reinstatement. Shortly after this, he was killed. Evidence was introduced that before the insured's death the application for reinstatement had either been approved or practically approved at the home office. No notice of the acceptance of the application for reinstatement was given to the insured during his lifetime. The court held that under such circumstances no reinstatement took place.

MISREPRESENTATION OF AGE:—(Home Life Insurance Company *vs.* Zuribowitz, Supreme Court of Rhode Island, 87 Atl. Rep. 225.) The insured in this case gave his date of birth in the application as 1856. It was shown that when he was naturalized, when he registered as a voter, and in an application to the New York Life, he stated different dates, each of which made him four or more years older than that in the application, that is, made him

age 60 or over. It was also shown that at such ages the company's medical requirements were more stringent than at the age as given by the application. The action by the company was to cancel the policy, and the decision in the lower court was in its favor. On appeal this was affirmed, the court saying in the opinion:

"We think that it is quite apparent that this statement of age in the application here in evidence was false and fraudulent, and intended to conceal his true age, for the purpose of obtaining the insurance at a lower rate than he was entitled to. If he did not know his age, he should have so stated. The company was entitled to know his age, as an important element in the risk involved, and in view of their rules, not only regarding the rate of premium, but also as to the limitation of amount of insurance of persons over 60 years of age, and as to the more stringent medical examination required in such cases."

While the court stated that according to the finding of the lower court the defendant had also misrepresented his condition of health, yet the wording of the opinion would lead to the belief that the decision was chiefly on the basis of the fraudulent misrepresentation of age.

This is important, since it has been questionable whether, with the provisions of the policies now issued, that if the age of the insured has been misstated the amount payable should be that which the premium would have purchased at the correct age, a fraud in the statement of age could be availed of as a defence. The holding of the present case is, apparently, that it can, and such result is the equitable one where a misstatement has been deliberate, since frequently the age enters into the question of the acceptability of the risk, companies becoming more careful as the age advances even where there are no specific rules regarding it, and, furthermore, under a misstatement the company always runs the risk that the misstatement will not be discovered.

Misleading Policy Form:—(Nellis *vs.* Western Life Indemnity Co., Court of Appeals of New York, 100 N. E. Rep. 1119.) The insured, Nellis, was originally a member of a fraternal order and held two certificates each for $1,500. The fraternal order was later reinsured in the defendant company, and at the time of such reinsurance Nellis received the policy in suit, which on its face purported to be for three thousand dollars and called for a monthly

premium of $3.00 per $1,000 insured. On the third page of the policy, however, was printed a blank form of application in which were included certain warranties and also an agreement containing various provisions in fine print. One of the provisions was to the effect that the insured consented to a lien against the policy to the amount of "the net single premium as determined by the Actuaries or Combined Experience Table of Mortality and interest at four per cent." Not only was the insured's attention at no time directed to this lien, but the secretary in persuading him to accept the policy in the new company referred to the contract as giving $3,000 insurance at a monthly premium of $9. When the policy became a claim by death the company insisted that it was entitled to deduct $1,950 with interest from the face amount of the policy, leaving a liability of but $450.

The court in affirming a verdict in favor of the plaintiff for the full amount said, "Where a person accepts insurance upon terms so written in the policy by the insurer, either intentionally or otherwise as to be calculated to deceive, and in such ambiguous language that it is possible to construe them in more than one way, a construction in favor of the understanding of the insured at the time the policy was taken should be sustained."

This case is to be distinguished from those where the clear purport of the policy contract is disregarded, since here the fact that the amount insured was not the $3,000 shown on the face of the policy could apparently be ascertained only by the most careful inspection and analysis of inconspicuous clauses.

KNOWLEDGE OF AGENT:—(Mallen *vs.* National Life Insurance Association, Kansas City Court of Appeals, 153 S. W. Rep. 1065.) The function of the soliciting agent is that of getting new insurance, connected with which is the delivery of the policy when issued. His real and apparent authority are strictly limited but there has been some tendency by certain courts to extend this. This makes the present case the more important, as in it the court recognizes that the power of the agent does not extend beyond his real or apparent authority. It is notable that the decision is in a state some of whose laws and decisions bear heavily on life insurance companies.

The applicant stated to the medical examiner that he had not had a certain disease, because of which some six months after the policy

was issued an operation was performed of which he died. It was sought to avoid the effect of the false representation on the ground that the soliciting agent had been informed of the fact, and had said that it made no difference, and requested the applicant to answer "No" to the question when asked by the examiner. The court in reversing a judgment for the plaintiff said:

"In this case it was not the duty, nor within the authority, real or apparent, of the soliciting agent, to receive information or make inquiry as to the character of diseases. The defendant company did not seek information of deceased's health through the solicitor. That authority was placed with the medical examiner, as deceased must have known; for it was before him his examination took place, and his answers were written down and signed by him. Deceased knew he was not informing the company as to his disease; for he consented to the soliciting agent to suppress that information, and to deny the existence of the disease to the men selected by the insurer to find out the facts."

(See Legal Notes, *T. A. S. A.*, XII, 91.)

Power of Commissioner to Refuse License:—(State *ex rel* Phoenix Mutual Life Insurance Co. *vs.* McMaster, Supreme Court of South Carolina, 77 S. E. Rep. 401.) This case was for an application for a writ of mandamus compelling the issuance of a license. The statute of the state regarding the furnishing of a bond or depositing of securities read as follows:

"Before licensing any insurance company to do business in this State, the Insurance Commissioner shall require each such company to deposit with him an approved bond or approved securities, in the discretion of the Commissioner, as follows: Each legal reserve life insurance company, twenty thousand dollars; each fire, accident or casualty or surety company, ten thousand dollars."

The company tendered a bond, but the commissioner required an affidavit showing that at least one-quarter of the reserve on South Carolina policies had been invested in the securities named in the act in question, which were all South Carolina securities. The company contended that it was beyond the power of the insurance commissioner to refuse to issue a license unless it invested one quarter of the reserve on South Carolina policies in the South Carolina securities named. The court, in deciding against the company, avoided this point by holding that the commissioner had the power

to refuse to accept the bond tendered, and that the company did not actually tender any securities in place of such bond.

Does the Power to Regulate Insurance in Canada belong to the Dominion Government or to the Separate Provinces? —(Reference re Insurance, Supreme Court of Canada. Not yet reported.) For many years the Dominion government has assumed that the regulation of insurance was within its domain, and Parliament has passed statutes of regulation. In Canada certain powers are enumerated as belonging to the Dominion government and certain others are enumerated as being reserved to the Provinces, but the residuary governmental powers are reserved to the Dominion government. In the present reference the question is, whether the Dominion Parliament has power to regulate insurance. The enumerated power of the Dominion under which the regulation of insurance would probably come, if under any, is the power "to regulate trade and commerce." If it does not come under this and is still within the domain of the Dominion government, it would come under the residuary powers. On the other hand, it was claimed that insurance comes under one of the subjects specifically reserved for legislation by the Provinces, namely, "Property and Civil Rights in the Provinces." The court was divided, the majority holding that the statute of the Dominion Parliament attempting to regulate insurance was *ultra vires*. In the opinions much attention was given to the question whether insurance came under the heading of "trade and commerce" and the minority held that it did.

The importance of the question whether the Canadian regulation of insurance shall be by the Dominion government for all Canada, or by each of the separate Provinces independently, is too self-evident to comment upon. The question, so far as it turns on the further question of whether insurance comes under the heading of trade and commerce, is similar to that before us in the United States as to whether the Federal government has power over the business of insurance carried on in a state by companies organized in other states as being interstate commerce.

The importance of the decision itself is lessened by the fact that there is the opportunity of further resort to the Privy Council in England. This will probably be availed of, and the final decision will come from it in due course of time.

MATERIALITY OF REPRESENTATION; POWER OF SOLICITING AGENT TO BIND THE COMPANY:—(Gardner *vs.* North State Mutual Life Insurance Co., Supreme Court of North Carolina, 79 S. E. Rep. 806.) When the application was taken a binding receipt was issued which provided that, in case the application was approved, the insurance should relate back to and take effect as of the date of the receipt. In the application, the applicant stated that he had not been intimately associated with any one suffering from any transmissible disease within a year before the application. As a matter of fact, he had shortly before nursed members of his family in typhoid fever. When the policy was delivered he was himself ill of typhoid of which he shortly afterwards died. The court in discussing the materiality of the point falsified used these words:

"It may be stated as a general rule that where, in an application for insurance, a fact is specifically inquired about, or the question is so framed as to call for a true statement of the fact, or to elicit the information desired, reason and justice alike demand that there should be a fair and full disclosure of the fact or at least a substantial one."

It was claimed that the delivery of the policy while the insured was ill with typhoid was a waiver. If neither the agent nor the company knew of the false statement in the application it could not have acted as a waiver because, as the court said, "There can be no legal waiver of a right without a knowledge of the right which is claimed to have been relinquished." Discussing the possibility of the agent's knowing but still delivering the policy the court said:

"But the other party (here Gardner) must have had knowledge of the principal's right, and that the agent was defrauding his principal, or was disobeying instructions, or acting without the scope of his employment, or he must have colluded with him, and thereby obtained something belonging to the principal, without being legally entitled thereto."

Abstract of the Discussion of Papers Read at the Previous Meeting.

Mortality Surplus and Its Distribution—Henry Moir.

Vol. XIV, page 17.

Written Discussion.

Mr. Gore:

Mr. Moir's ingenious analysis in Tables V and VI shows that surplus arising from sub-tabular mortality is not at all likely even to approximate a percentage, constant for all ages, of the "cost of insurance" as measured by the American Experience Table. I can hardly agree with Mr. Moir, however, that even "in theory it is correct to distribute mortality profit, when dealt with separately, as a function of the cost of insurance on the reserve basis," as the fact that the mortality actually experienced differs from that according to the valuation table proves that the latter is an inaccurate measure, but obviously does not prove that the inaccuracy is a constant percentage of the tabular rate at all ages. Many companies value part of their issues by the Combined Table and the remainder by the American, and it is clear that if the actual mortality were a constant percentage of that expected by one of these tables, it would be a varying percentage of the expected by the other. It is evident, therefore, that, if the principle be accepted that the distribution of surplus is to be governed by the origin thereof, a table, concerning which we merely know that it does not accurately represent the actual mortality, has no theoretical claim to be accepted as the basis of distribution of mortality surplus.

At first sight it might appear that the distribution of mortality surplus as it arose would be the proper and most convenient course. This would provide considerable amounts at the younger ages and early durations, diminishing with increasing age, and, for some years after entry, with duration. If this course were followed, however, it would be only proper that all other surplus be distributed just as it arose, which would mean, in the case of a net level premium valuation, a substantial negative dividend for the first year, on account of the loss from loading. To avoid this, in practice, the aggregate profit from loading is generally spread, more or less evenly, over the whole period of the insurance, and there is no harm done in similarly equalizing the allotment of the aggregate profit from mortality. This latter arrangement has the advantage of diminish-

ing the strain put upon the funds by the necessity for providing full reserves for new business. Mr. Moir has assumed that the mortality profit of the first five years will be used in repayment of the special cost of new business, but, as a matter of fact, those companies which distribute mortality surplus separately, ordinarily apply no portion thereof as an offset to the expenses of new business. Moreover, companies using the two-factor method of distributing surplus are, in effect, applying only a part of the early mortality surplus to relieve the burden of initial expenses.

If we suppose a company to have been in business so long that it has reached a position of relative equilibrium, i. e., that the distribution of the insurance in force according to age and duration remains practically unchanged from year to year, it is evident that any system of dividend distribution will be fair as between old and recent entrants, though, perhaps, not so as regards the insured under different plans and entry ages. For it is clear that the issues of any given year will pass through the same stages as those of every other year. Whatever advantage they may gain as new business, at the expense of the then older policies, they will have, in turn, to contribute to the successive new issues of subsequent years. A sudden expansion in new business, however, would disturb the equilibrium and would probably inflict some loss upon the entrants prior to and during the early years of the new period of expansion, supposing a net level premium valuation to be made and a two-factor or three-factor surplus distribution method to hold.

A very curious result, on the supposition of a company writing a substantially increasing new business each year, came to light in connection with some calculations I had made some time ago.

A hypothetical company was assumed to be writing twenty-year endowment insurances at age at entry thirty-five. Endowment insurances were selected because the complete history of a year's issues lasted for only twenty years. The gross premium was assumed to be $51.91 per $1,000, the actual rate of interest four per cent., the experienced mortality that shown by the New York Life Compound Progressive Table, and the valuation American three per cent. Rates of expense, surrender, and surrender allowances were all in fairly close accord with actual experience.

It was found that it was possible to return as dividend the amount of excess interest on reserve—one per cent.—plus a uniform 39.7 per cent. of the loading. On this dividend scheme, however, the net accumulations in respect of a single year's business would never reach the reserve till just at maturity. Supposing the company to be writing a uniform business, and to be more than twenty years old, there would always be just twenty years' issues on the books, and the amount by which the actual accumulations fell short of the reserve would be constant. If, now, we suppose this deficiency to have been made good at some time in the past, not only would no further additions thereto be required, but the interest on the added fund would be available to increase the dividends. In the calcula-

tions made, it appeared that the return of loading would be increased to 45 per cent. per annum, so that every policyholder would receive as dividend every year 5.3 per cent. of the loading, or just about one per cent. of the premium, more than his own contributions could provide, while the new business remained constant. When new business increased each year by 10 per cent. of the issues of the previous year, however, a very different state of affairs was disclosed. The payment of excess interest as earned, and a uniform loading return of 39.7 per cent., would result, as before, in accumulations less than the reserves, but (after twenty years) each item of the company's financial affairs would grow 10 per cent. per annum, and the deficiency would form no exception. It follows that if, from any source, the deficiency were made up as at the beginning of a year, the fund so provided increasing only by 4 per cent. on account of interest, there would remain 6 per cent. of the initial amount to be provided, with the result that of the loading surplus of 39.7 per cent., only 30 per cent. could be distributed. This means that every policyholder would suffer an annual loss of 9.7 per cent. of loading, or about 2 per cent. of his premium, on account of the dividend system not being properly co-ordinated with the valuation basis. In practice, of course, no company could keep up a perpetual 10 per cent. increase in new business, but it seems evident that a policy of rapid expansion in new business should be accompanied by very careful scrutiny of the effect on dividends.

Paradoxical as it may appear, our hypothetical company may continue to write its increasing business, maintain its full reserves, and thus distribute the same amount of cash dividends each year, and yet give to each policyholder the full earnings under his policy. Our calculations showed that this result would be almost exactly achieved by distributing the whole surplus as if it arose from excess interest, i. e., in proportion to reserve. This would give a very rapidly ascending scale of dividends, whose value at the inception of the policy would be precisely equal to that already mentioned as practicable, namely, 1 per cent. on reserves plus 39.7 per cent. of loading.

My object in presenting the foregoing statement is to show that in the case of a progressive company the two-factor method is apt to err in the direction of making the early dividends too large rather than too small. As the introduction of a separate mortality factor would increase the relative amount of early dividends, the error would be accentuated, and I therefore agree with Mr. Moir that it is usually preferable not to distribute the mortality surplus separately but rather to merge it with surplus arising from loading margins and sundry sources.

MR. ARTHUR B. WOOD:

The main question considered by Mr. Moir in this paper is whether the method of distributing mortality profit as a uniform

percentage of the cost of insurance produces equitable results when consideration is given to the influence of the age at entry and the duration of the assurance upon the actual mortality.

For the purpose of illustrating the general trend of the mortality profit the new M.A. Table is taken to represent actual experience, and the resulting profit on the basis of American 3 per cent. net premiums and reserves is compared with 20 per cent. of the cost of insurance. The mortality gains during the first five years being regarded as applied against initial expenses, only policies more than five years in force are dealt with. On this assumption there would, of course, be no allotment of mortality profit until the sixth policy year. It is apparent from Tables V and VI that a uniform percentage of the cost of insurance unduly favors the older ages at entry and the policies of long duration, a result agreeing with our preconceived ideas. These tables, however, place the facts before us in an impressive manner. It may be observed that there is no necessary correspondence between the actual figures compared in these tables. In other words, the actual mortality profit on the basis assumed does not necessarily correspond to 20 per cent. of the cost of insurance by the American Table. In fact it is not unlikely that in a company experiencing M.A. mortality and transacting a considerable volume of new business, the mortality gains would average considerably more than 20 per cent. Instances may be found in actual practice where the mortality factor used in the distribution is even as high as 35 per cent. If a larger percentage than 20 were used in the comparison, the discrimination in favor of the older policies and the advanced ages at entry would be even more marked. This would also be the case if the actual mortality rates based on amounts assured should prove to be higher at the older ages than the M.A. experience, as is not unlikely.

But in order to arrive at any satisfactory conclusions as to whether one system of profit distribution is more equitable than another, we must consider the method in its entirety and not limit our criticism to one particular factor, for it is the final cash dividend with which we are chiefly concerned rather than its individual component parts. Mr. Moir's paper really brings up for consideration the relative merits of the so called "three factor" and "two factor" methods. In contrasting the two methods it is unnecessary to comment upon the gains from surrenders and lapses, or the profit from interest, as both deal with these items in the same way.

The "three factor" method involves the assumption that expenses are distributed uniformly over all premiums, and that mortality profit arises equally in proportion to the cost of insurance irrespective of the age at entry or the duration of the assurance. Both assumptions we know to be contrary to the facts. The effect is that the older ages at entry and the older policies do not receive a sufficiently large proportion of the loading profit, but on the other hand, as the present investigation clearly shows, they receive more than their share of mortality profit. It may be argued, therefore,

that these two influences tend to counteract one another and to produce a final cash dividend which gives equitable results on the whole. It is evident, however, that the excess of the mortality profit at the advanced ages far exceeds any possible saving in expenses. Thus, a policyholder aged 70 who insured at age 30 on the whole life plan would receive $4.00 per thousand more profit from mortality than his policy had earned during that particular year, which is about 17½ per cent. of an annual premium of $22.85. On the other hand, a policyholder aged 40 who insured at age 30 would receive $1.90 less than the actual mortality profit earned in the tenth policy year; whereas the saving in expenses under a policy forty years in force, as compared with one ten years in force, would be merely the renewal commission of not more than 7½ per cent. Therefore, when account is taken of the incidence of both expense and mortality, the "three factor" method appears to favor the older policyholders unduly. Mr. Moir has referred to the practice followed by some companies of using a decreasing scale of percentages of the cost of insurance. This is a recognition of the fact that a more equitable basis for the distribution of mortality profit is desirable.

Under the "two factor" method on the other hand, the mortality gains, as a whole, are regarded as the direct result of the expenditure in connection with the acquisition of new business. The mortality profit is, therefore, applied as an offset to expenses, and the net surplus derived from all sources, except interest, is distributed in proportion to the loadings. The principle of applying mortality gains as an offset to expenses is recognized in the select and ultimate method of valuation embodied in the New York law, and if it is correct to treat the mortality gains during the first five years in this manner, it is equally logical in the distribution of profits to regard the total mortality gains in the same light. As this subject has been thoroughly discussed by the Society on previous occasions, I will not deal with it at greater length, but will limit my remarks to certain criticisms of the "two factor" method.

The objection has been made that it unduly favors the younger ages at entry, because of the higher proportionate loadings at the older ages. This may be a valid criticism, as far as the question of the distribution of expenses is concerned, but, on the other hand, it must be remembered that as the age at entry advances the ratio of the actual mortality profit to the loading decreases. The combined effect of expense and mortality must, therefore, be considered before passing judgment.

To form some conception of the relative value of the mortality profit at different ages at entry and to compare it with the loadings, I have made certain calculations on the basis assumed by Mr. Moir in his paper, viz., M.A. experience and American 3 per cent. reserves, the results of which are given below. The present value of the mortality gains is shown separately for the first five years and for the subsequent years, and the mortality gains after the first five

years are compared with the loadings on the premiums payable thereafter. To simplify the comparison, the present value of the mortality gains is distributed uniformly over the future premiums. In the case of the 20 payment life plan, the mortality gains for the years 6–20 inclusive are distributed equally over the last fifteen years of the premium paying term; the annual equivalent of the mortality profit after twenty years is also shown.

WHOLE LIFE PLAN.

Age at Entry.	Present Value of Mortality Gains.			Gross Premium.	Loading.	Annual Equiv. after Five Yrs. of Subsequent Mortality Gains.	Percentage of Loading.
	Years 1-5.	After 5 Years.	Total.				
25	16.74	45.11	61.85	20.14	4.03	2.40	60%
30	18.29	41.68	59.97	22.85	4.57	2.36	52
40	18.42	25.37	43.79	30.94	6.19	1.72	28
50	14.57	12.61	27.12	45.45	9.09	1.14	12

TWENTY PAYMENT LIFE PLAN.

Age at Entry.	Present Value of Mortality Gains.				Gross Prem.	Loading.	Annual Equiv. of Mort. Gains.		Percentage of Loading.
	Years 1-5.	Years 6-20.	After 20 Yrs.	Total.			Years 6-20.	After 20 Yrs.	Years 6-20.
25	16.28	16.84	8.15	51.27	30.12	5.14	2.67	.93	52%
30	17.19	25.93	5.36	48.48	32.87	5.68	2.60	.70	46
40	17.92	18.71	1.43	38.06	40.38	7.24	1.94	.28	27
50	14.16	11.03	——	25.19	52.87	9.92	1.23	—	12

On the whole life plan, the mortality profit after five years is equivalent to 60 per cent. of the annual loading at age 25 at issue, 50 per cent. at age 30, 28 per cent. at age 40, and 12 per cent. at age 50. On the twenty payment life plan, the percentages are 52, 46, 27, and 12.

Incidentally it may be remarked that the present value of the mortality gains during the first five years, on the assumption of M.A. mortality, would permit of a considerably higher expense allowance for new business than that fixed by the New York law, except at the older ages.

It is apparent that the use of but one factor for the distribution of the net profit from loading and mortality combined, produces results, as between different ages at entry, similar to what would follow from the application of the "three factor" method if modifications were introduced to allow for the incidence of both expense and mortality, that is to say, if the loading profit were distributed by a graded scale of percentages increasing according to age at entry, and the mortality profit by a decreasing scale of percentages of the cost of insurance. It is also evident that a similar effect is produced as between policies of different durations taken out at the

same age. In other words, the uniform profit for each age at entry may be regarded as composed of two parts, viz., loading profit and mortality profit, the one increasing and the other decreasing with the duration of the assurance. In fact, this method probably conforms more closely to the actual conditions of the business than the "three factor" method as commonly applied.

Mr. Moir has referred to the objection raised in regard to limited payment life policies, that they receive no profit from mortality after they have become paid-up; as against this can be advanced the argument that, as there is no loading, no charge for expenses is made against paid-up policies, and that they receive as high a rate of interest profit as premium paying policies. The annual mortality profit under paid-up policies will apparently average considerably less than $1.00 per thousand of insurance, and this may fairly be regarded as an offset to the expense of handling these policies.

Personally I cannot see that any advantage is gained by attempting to introduce too great refinements into the distribution of profits. There are so many factors which affect the problem that it is not practicable to give effect to them all, and it seems better, therefore, to regard the matter from the broad general standpoint. It is, of course, desirable that the method adopted should be capable of defence on theoretical grounds, as well as of those of practical expediency, but the simpler the method the better. Whether one system is more equitable than another depends entirely upon the point of view, and a good argument can be made in favor of almost any method which gives results that are satisfactory to the general body of the policyholders. To best attain this end the aim should be to produce a scale of dividends increasing both according to age at entry and duration of insurance. Each of the methods under consideration accomplishes this. The "two factor" method, however, has the advantage of simplicity, and even from the standpoint of equity, it probably gives as satisfactory results as another method which attempts to follow the actual conditions of the business more closely. My views are in accord with the belief expressed by Mr. Moir that, when due consideration is given to the many elements which affect the problem, we probably secure a more equitable distribution by merging the mortality profit with the general earnings and making the allotment in proportion to the loadings than by allotting the mortality gains as a percentage of the cost of insurance.

MR. PAPPS:

Mr. Moir has raised some interesting questions regarding the distribution of mortality surplus. The fundamental principle of the contribution plan of distributing surplus, as I see it, is that we should refund the surplus in accordance with the sources from which it has arisen. It is true that the larger mortality savings on new business are not now all directly returned as first year dividends, for a considerable portion is used as an offset to the necessarily

heavy expenses incidental to writing new business. This may be considered as a departure, to a limited extent, from the principle of the contribution plan, the strict compliance with which would necessitate much larger expenses charged against new policies, and a much larger credit on account of favorable mortality.

Mr. Moir advocates throwing all mortality savings in with surplus earnings other than interest savings, and distributing this general surplus in proportion to the loading. He suggests that, since there is a general tendency for expenses to decrease with the duration of the policy, causing a general increase in the savings from loading, and also a tendency toward a decrease in profits from mortality as the duration increases, the one may be considered as an offset to the other. This appears to be the justification for distributing both the savings from loading and the savings from mortality in proportion to the loading. While the expenses of the early years are very much greater than when a policy has been a considerable length of time in force, it is difficult to see how there can be any material decrease in expenses or increase in savings from loading after a policy has been fifteen years in force. Table V, which appears on page 21 of Mr. Moir's paper, shows a decided decrease in the mortality profits after a policy has been more than fifteen years in force. It is apparent, therefore, that the increasing savings from loading cannot keep pace with the decreasing savings from mortality.

It may be mentioned also that since the loading is generally increased with the age at issue, it follows that if the mortality saving is to be distributed on the basis of loading, the mortality profits will be increased for policies issued at older rather than at younger ages. Mr. Moir's Table V, already referred to, shows that, on the whole, the mortality profits are smaller at the older than at the younger ages.

In view of the foregoing considerations it is difficult to see the advantage of using the two factor instead of the three factor method, so far as the equitable distribution of surplus is concerned.

I was much interested in the suggestion of Mr. Henderson, referred to by Mr. Moir, that the mortality surplus should be distributed as a constant function of the amount at risk. In this connection it may be mentioned that the heading of Table VIII, on page 23 of Mr. Moir's paper, refers to the cost of insurance, whereas it should refer, as I understand it, to the amount at risk. It is interesting to see how Mr. Henderson's suggestion might work out in practice, and with this object in view I first prepared for the ordinary life plan for selected ages and durations what may be termed "standard dividends." Although the dividend scales of the representative companies differ materially, I do not believe that there is the same difference in the actual dividends which a comparison of the scales might lead one to suppose. In arriving at what I have called the standard dividends, I have assumed that

the company is operating on the American 3 per cent. reserve basis and is charging the premiums referred to on page 24 of Mr. Moir's paper. I have assumed that the dividend is made up of three factors: first, $1\frac{3}{4}$ per cent. of the initial reserve; second, 15 per cent. of the cost of insurance, and third, the margin of loading. This latter factor is arrived at by deducting from the loading 15 per cent. of the gross premium, and improving the difference with interest for one year at the rate of $4\frac{3}{4}$ per cent. These dividends are shown in Table D.

The next step was to assume that there was a certain amount of business in force for each age and duration shown. The amounts assumed to be in force were chosen with the idea that in the aggregate they would represent the ordinary life business of a well-established company. The amount of business assumed to be in force, amounting to $100,000,000 in the aggregate, is as follows:

TABLE A.

DIVISION OF $100,000,000 OF BUSINESS.

Year.	Age 25.	Age 45.	Age 65.
1	$ 6,000,000	$23,400,000	$ 600,000
5	8,750,000	25,550,000	700,000
10	4,500,000	10,350,000	150,000
15	2,800,000	5,120,000	80,000
20	2,000,000	3,000,000	——
30	2,250,000	2,750,000	——
40	1,000,000	1,000,000	——
	$27,300,000	$71,170,000	$1,530,000

The next step was to ascertain the amount of mortality surplus which would be distributed according to the standard dividends previously computed. The amount of such surplus is as follows:

TABLE B.

MORTALITY SURPLUS ACCORDING TO STANDARD DIVIDENDS (15 PER CENT. OF COST OF INSURANCE).

Year.	Age 25.	Age 45.	Age 65.
1	$ 7,200.00	$ 38,376.00	$ 3,468.00
5	10,412.50	45,223.50	5,409.00
10	5,355.00	21,321.00	1,239.00
15	3,388.00	12,748.80	748.00
20	2,500.00	9,270.00	——
30	3,555.00	12,677.50	——
40	2,380.00	5,790.00	——
	$ 34,790.50	$145,406.80	$10,864.00
	145,406.80		
	10,864.00		
	$191,061.30		

The next step was to set out the amount at risk for each age and duration based upon the selected amount of insurance assumed to be in force. The total mortality surplus amounting to $191,061.30 was then distributed in proportion to the amount at risk. The mortality surplus thus distributed according to the net amount at risk is as follows:

TABLE C.

MORTALITY SURPLUS ACCORDING TO AMOUNT AT RISK ($2.23 PER THOUSAND).

Age.	Yr.	Terminal Reserve.	Amount at Risk per $1,000.	Amount in Force.	Amount at Risk.	Mortality Profit per Group.	Mortality Profit per Policy.	Dividend.
25	1	$ 8.60	$991.40	$ 6,000,000	$ 5,948,400	$ 13,274.88	$2.21	$ 3.55
	5	45.76	954.24	8,750,000	8,349,600	18,633.58	2.13	4.10
	10	98.94	901.06	4,500,000	4,054,770	9,048.92	2.01	4.89
	15	160.36	839.64	2,800,000	2,350,992	5,246.65	1.87	5.79
	20	230.50	769.50	2,000,000	1,539,000	3,434.54	1.72	6.84
	30	394.11	605.89	2,250,000	1,363,252	3,042.33	1.35	9.29
	40	570.12	429.88	1,000,000	429,880	959.35	.96	11.98
					$24,035,894	$ 53,640.25		
45	1	$ 19.61	$980.39	$23,400,000	$22,941,126	$ 51,197.09	$2.19	$ 4.65
	5	102.20	897.80	25,550,000	22,938,790	51,191.88	2.00	5.88
	10	212.62	787.38	10,350,000	8,149,383	18,186.76	1.76	7.55
	15	327.27	672.73	5,120,000	3,444,378	7,686.73	1.50	9.28
	20	441.35	558.65	3,000,000	1,675,950	3,740.17	1.25	11.04
	30	646.62	353.38	2,750,000	971,795	2,168.73	.79	14.24
	40	817.34	182.66	1,000,000	182,660	407.64	.41	16.89
					$60,304,082	$134,579.00		
65	1	$ 39.87	$960.13	$ 600,000	$ 576,078	$ 1,285.62	$2.14	$ 8.46
	5	193.29	806.71	700,000	564,697	1,260.22	1.80	10.85
	10	367.43	632.57	150,000	94,886	211.75	1.41	13.58
	15	526.88	473.12	80,000	37,850	84.47	1.06	16.06
	20	673.03	326.97	——	——	——	.73	18.33
	25	802.11	197.89	——	$ 1,273,511	$ 2,842.06	.44	20.39
					$85,613,487	$191,061.31		

$$\frac{191,061.30}{85,613,487} = \$2.2316729$$

Comparing Tables B and C, it will be seen that Mr. Henderson's suggestion results in increasing very largely the mortality surplus distributed to policies at the younger ages at entry and shorter durations. If our present dividends are reasonably correct, although our dividend scales may differ widely, it seems to me that we can hardly adopt Mr. Henderson's suggestion for distributing mortality surplus unless we at the same time change materially the manner in which we shall assess expenses. Were the expenses to be assessed more nearly in accord with the manner in which they are incurred, it might well be that at least a large part of the difference in the distribution of the mortality surplus arising from Mr. Henderson's suggestion would be offset by the change in the assessment of the expenses.

It may be of interest to compare the mortality factor of the dividend as well as the dividend itself, according to the standard dividend scale above referred to, and according to Mr. Henderson's suggestion. This is done in the following table:

TABLE D.

Age.	Year.	Standard Dividend Scale.		Mr. Henderson's Suggestion.	
		Mortality Factor.	Dividend.	Mortality Factor.	Dividend.
25	1	$ 1.20	$ 2.54	$2.21	$ 3.55
	5	1.19	3.16	2.13	4.10
	10	1.19	4.07	2.01	4.89
	15	1.21	5.13	1.87	5.79
	20	1.25	6.37	1.72	6.84
	30	1.58	9.52	1.35	9.29
	40	2.38	13.40	.96	11.98
45	1	1.64	4.10	2.19	4.65
	5	1.77	5.65	2.00	5.88
	10	2.06	7.85	1.76	7.55
	15	2.49	10.27	1.50	9.28
	20	3.09	12.88	1.25	11.04
	30	4.61	18.06	.79	14.24
	40	5.79	22.27	.41	16.89
65	1	5.78	12.10	2.14	8.46
	5	6.87	15.92	1.80	10.85
	10	8.26	20.43	1.41	13.58
	15	9.35	24.35	1.06	16.06
	20	10.37	27.97	.73	18.33
	25	11.75	31.70	.44	20.39

It will be noticed that the effect on the distribution of mortality surplus has been measured only for the ordinary life plan. This is, of course, the plan where the largest amount is at risk and where, in consequence, the greatest difference in the mortality surplus might be expected to occur. At the same time there would in practice doubtless be a shifting of the amount of the mortality surplus distributed as between various plans as well as between ages and durations.

ORAL DISCUSSION.

MR. DAWSON: I intended to prepare a short discussion of this paper, but I was not able to do so, and consequently I will make a few brief statements, which perhaps will answer the same purpose.

I was very greatly impressed with the general correctness of Mr. Moir's paper, and with its great usefulness as a means of opening a very valuable discussion which I hope will not cease with this meeting of the Society. But one point which he made seems to me not to be well taken, namely, that there is any especial significance in the simple fact that there is a decrease in mortality savings and an increase in the savings from loading. Unless those two things can be shown to be inter-related, so as to correspond in some way,

they appear to me to be merely incidental; and an argument in favor of the two-factor method, which I myself have urged upon the members of the Society as preferable, could not, I think, safely be based upon this. The true basis, it appears to me, is that a distribution method should be designed along lines which have been presented here, which should be so designed from the start as fairly and approximately to represent the contribution method.

That idea has been worked out in other countries, where they have made a study of participation, with a view, not merely to distribute in accordance with the contributions after they have realized, but in connection with methods of anticipating what those gains will be and representing them in the initial calculations before policies are issued.

In the United States, in connection with annual dividend policies, it appears to be entirely settled that a desirable thing is a dividend composed of two factors, one of which shall be as nearly constant as possible, and the other increasing, so that the company may show a substantial dividend virtually from the start, slightly increasing, and, if its experience will warrant, constantly increasing. It is entirely true that by treating the mortality as an offset to the cost of new business, according to the principles of the select and ultimate method, which principles were known in insurance practically before this method of valuation was brought out, an approximation of this desirable standard is attained. Several of the companies which specialized in annual dividend policies—and if it were not for the rule of the Society that prevents the mentioning of individual companies, I could mention them—practically adopted that idea in fixing the mortality factor in the dividend calculations. They approximately fixed their mortality factor in dividend distribution at a point which represented the gains on mortality on their business *after* it had reached ultimate mortality, leaving the special gains of early years of insurance, due to fresh medical selection, to be treated, to all intents and purposes, as an offset to the expense of new business.

Now, the whole problem of reconciling this desirable form of distribution with the principles of contribution, which all good actuaries the world over, even where the name scarcely is known, have recognized and thoroughly accepted, is chiefly one of determining what this constant factor in calculating dividends should be.

The increasing factor in dividend distribution, except in a few cases, has usually been provided merely by allowing the excess interest earned upon the investment of the reserve of the policy; and as nearly all policies have increasing reserves, the result has been increasing dividends when the interest rate was stable.

So the only thing which requires to be carefully studied in order that the mortality factor shall be properly taken into account, is how the constant dividend factor may be determined. I am confident that careful study of the subject by the members of this

Society will lead them to the belief that this should be brought about by making proper calculations of premiums at the outset in accordance with what is fairly and actually to be expected in regard to mortality, in a manner somewhat similar to that which has been used by British actuaries for a long time in the calculation of their premiums in reference to uniformity of compound or simple bonuses.

MR. WALTER C. WRIGHT: It struck me as I listened to the different arguments that they all erred in one particular. They magnify the importance of what seems to me a necessarily misleading consideration, which has been, as I have noted, considerably discussed of late, and that is the incidence of expense, and, in the same way, the incidence of gain from mortality, which is not the real fundamental important thing. The fundamental thing is the gain itself and the interest of the parties who share in it.

MR. MOIR.

(AUTHOR'S REVIEW OF DISCUSSIONS.)

There is one point in my paper which apparently I did not make sufficiently clear, although I did state it, namely, that I did not wish to dogmatize on anything, especially on this subject; because I know that there are so many different methods of distribution, and so far as I have studied them, most of them reasonably fair. The one-factor method is used by one company, and the results seem good. The two-factor method is used satisfactorily by many companies, and in one of them no loading factor is used, but only interest and mortality, and the result there seems to be good. The three-factor method is used by many companies, and that also seems to work well. Since all three methods are in general use, and all give fair results to policyholders, it is apparent that the problem is being solved by different actuaries in different ways, each equitable and fair. Accordingly, surplus distribution is not a subject on which any one can dogmatize. The paper was written solely in order to get a little more light on mortality surplus. I had no thought when I started the paper of making any reference to the two-factor method. That came as an after-thought, and more emphasis has been laid on it in the discussion than it deserves. I have analyzed the surplus as we seem to be earning it, to determine what part of the total surplus should be properly considered as mortality surplus according to the latest statistics available.

I have little to answer in the matter of criticism, because I find that the critics have answered one another, leaving me little to say. The question of the reduction of commission expense was referred to by Mr. Papps, but it seems to me there is such a notable decline in the expense charge after fifteen years, through agency commissions running out, and various other reasons, that I cannot understand Mr. Papps's difficulty.

MR. PAPPS: If I might interrupt, my intention was to carry out this idea after the fifteen years expired, and from that point on there would be little if any change in the loading.

MR. MOIR: I must have partly misunderstood Mr. Papps's remarks; but most companies are still working out old commission contracts, some running for twenty years and some even for the duration of the policies. Accordingly, until the old business (*i. e.*, business written before the Armstrong Law) is worked off, it is clear that there must be a diminution in expense long after the passing of the fifteen-year point.

As regards Mr. Wright's criticism, it seems to me that true mutuality consists in fairness and justice to all classes of policyholders, and the paper points the way towards stricter equity and therefore more harmonious mutuality.

There are three statements made by Mr. Gore which may be correct, but which I cannot concur in without much more thorough investigation. These three remarks are:

(1) "Those companies which distribute mortality surplus separately ordinarily apply no portion thereof as an offset to the expenses of new business";

(2) "Companies using the two-factor method are in effect applying only a part of the early mortality surplus to relieve the burden of initial expenses"; and

(3) "In the case of a progressive company, the two-factor method is apt to err in the direction of making the early dividends too large rather than too small."

All three depend upon the method of charging first year's expenses.

CONCERNING THE AMERICAN EXPERIENCE TABLE OF MORTALITY—
S. A. JOFFE.

VOL. XIV, PAGE 27.

WRITTEN DISCUSSION.

MR. HENDERSON.

In examining some old manuscript books which were saved from the fire last year and which had not previously been referred to for some time, I came across a version of the American Experience Table which throws considerable light on its method of construction. The book in which this table is contained has, in fact, two different versions of the Mortality Table. On pages 2, 3 and 4 of the book there is given a table headed "Homans' Mortality," the radix of which is 10,000,165, and on pages 24, 25 and 26 is given a table headed "Homans' Mortality as adopted by New York," the radix of which is 100,000. The l_x and d_x columns of this table are exactly the same as in the American Experience Table as at present used, but the rate of mortality is given to seven decimal places and was evidently not made from the l_x and d_x columns. I had first supposed from their relative positions that the table with the larger radix was the principal table and that the other had possibly been formed from it, but for reasons which will be given later, I have come to the conclusion that the table with a radix of 100,000 is the original table and that the other was formed from it. The reason why I feel certain that one table was formed from the other is that the column of the logarithms of d_x is exactly the same in the two tables, and this column evidently forms the connecting link between the two. The conclusion which I have arrived at is that the table with the larger radix was formed entirely from the column of log d_x, the natural number being taken out to two places further than in the American Experience Table and the column summed backwards to form the l_x column, the 165 which appears in the radix being the balance of the accumulated extra figures.

I have with me a copy of the table which I believe to be the principal table and to show the basis upon which the American Experience Table was constructed. There are certain relations between the various columns of this table which show the order in which they must have been calculated. In the first place, the logarithm of l_x to seven decimal places agrees exactly with the values of l_x which contain from five down to one significant figures, so that log l_x was evidently formed from l_x. The reverse is the case

with respect to log d_x which does not agree exactly with the natural number d_x, so that d_x was evidently formed from log d_x. The same thing applies to the relation between the mortality and the logarithm of the mortality. The successive differences of the l_x column are exactly equal to the values of d_x. Since we have found that the d_x column cannot have been formed from the l_x but was formed from log d_x, it follows that the l_x column was formed by successive subtraction of d_x. The difference between log l_x and log d_x is exactly the figure which appears in the column headed "log mortality," so that log d_x was probably formed by adding log mortality to log l_x, although, so far as this relation would indicate, the reverse process might have been followed and log d_x might have been the fundamental column, as I believe it is of the other table to which I have referred.

In deciding which of the two tables should be considered as the fundamental one, I was guided partly by the fact, proved by Mr. Joffe, that the values of r_x (reciprocal of q_x) for ages 70 and over could be represented by a series of numbers to three decimal places. I, therefore, compared the columns headed "log mortality" in the two tables with the series of figures given by Mr. Joffe, and found that in the one table they agreed precisely and in the other table they did not agree. I then tested the ages under seventy to see whether a similar relation held, and I found that it held for every age down to 10 inclusive.

It is true that for the three ages 24, 29, and 35 the figures given in this table for log mortality differed by a unit in the last place from the figures obtained from the r_x column, but it must be remembered that at every age up to 40 inclusive it was necessary to interpolate by proportional parts on account of there being six significant figures in the value of r_x, and the marginal tables in books of logarithms then in use were not always as complete as those now given. With these three exceptions, the value of log mortality agrees exactly to seven decimal places with the value of r_x to three decimal places or to 6, 5, and 4 significant figures. I think this conclusively shows, therefore, that this column of r_x which was formed for each individual age to three decimal places is the fundamental column from which the American Experience Table was constructed.

I have prepared a table of the values of r_x and of the first, second, and third differences, but I am so far unable to come to any definite conclusion with respect to the method whereby the successive values were determined.

It is interesting to observe that the rates of mortality to six decimal places quoted by Mr. Joffe from Mr. Fackler's paper were evidently formed by cutting down the rates of mortality given in this table. My reason for this conclusion is that in every case where this table shows a five in the last place, the next figure is

HOMANS' MORTALITY AS ADOPTED BY NEW YORK.

	l_x	d_x	Mortality.	log l_x	log d_x	log Mortality.
10	100 000	749	007 4906	5.000 0000	874 5187	874 5187
11	99 251	746	7 5174	4.996 7349	872 8016	876 0667
12	98 505	743	7 5451	993 4583	871 1244	877 6661
13	97 762	740	7 5731	990 1701	869 4416	879 2715
14	97 022	737	7 6013	986 8702	867 7562	880 8860
15	96 285	735	7 6304	983 5586	866 1083	882 5497
16	95 550	732	7 6609	980 2307	864 5104	884 2797
17	94 818	729	7 6936	976 8908	863 0210	886 1302
18	94 089	727	7 7294	973 5389	861 6852	888 1463
19	93 362	725	7 7682	970 1701	860 4903	890 3202
20	92 637	723	7 8096	966 7845	859 4117	892 6272
21	91 914	722	7 8541	963 3817	858 4783	895 0966
22	91 192	721	7 9021	959 9567	857 6980	897 7413
23	90 471	720	7 9534	956 5094	857 0636	900 5542
24	89 751	719	8 0081	953 0393	856 5707	903 5314
25	89 032	718	8 0663	949 5461	856 2225	906 6764
26	88 314	718	8 1284	946 0296	856 0327	910 0031
27	87 596	718	8 1946	942 4843	856 0141	913 5298
28	86 878	718	8 2656	938 9098	856 1818	917 2720
29	86 160	719	8 3417	935 3057	856 5625	921 2568
30	85 441	720	8 4239	931 6663	857 1790	925 5127
31	84 721	721	8 5126	927 9911	858 0530	930 0619
32	84 000	723	8 6087	924 2793	859 2152	934 9359
33	83 277	726	8 7129	920 5251	860 6891	940 1640
34	82 551	729	8 8265	916 7223	862 5115	945 7892
35	81 822	732	8 9511	912 8701	864 7469	951 8768
36	81 090	737	9 0883	908 9673	867 4483	958 4810
37	80 353	742	9 2396	905 0021	870 6544	965 6523
38	79 611	749	9 4063	900 9731	874 3908	973 4177
39	78 862	756	9 5889	896 8678	878 6376	981 7698
40	78 106	765	9 7886	892 6844	883 4035	990 7191
41	77 341	774	10 0076	888 4098	888 7400	000 3302
42	76 567	785	10 2493	884 0416	894 7342	010 6926
43	75 782	797	10 5188	879 5661	901 5317	021 9656
44	74 985	812	10 8226	874 9744	909 3071	034 3327
45	74 173	828	11 1657	870 2458	918 1317	047 8859
46	73 345	848	11 5571	865 3705	928 2189	062 8484
47	72 497	870	12 0028	860 3200	939 6020	079 2820
48	71 627	896	12 5153	855 0768	952 5191	097 4423
49	70 731	927	13 1036	849 6098	966 9999	117 3901
50	69 804	962	13 7762	843 8803	983 0095	139 1292
51	68 842	1 001	14 5370	837 8535	000 3282	162 4747
52	67 841	1 044	15 3880	831 4922	018 6724	187 1802
53	66 797	1 091	16 3375	824 7570	037 9417	213 1847
54	65 706	1 143	17 3967	817 6050	058 0731	240 4681
55	64 563	1 199	18 5753	809 9837	078 9190	268 9353
56	63 364	1 260	19 8839	801 8426	100 3437	298 5011
57	62 104	1 325	21 3333	793 1196	122 1783	329 0587
58	60 779	1 394	22 9374	783 7536	144 2970	360 5434
59	59 385	1 468	24 7152	773 6768	166 6402	392 9634
60	57 917	1 546	26 6909	762 8061	189 1688	426 3627
61	56 371	1 628	28 8775	751 0557	211 6157	460 5600
62	54 743	1 713	31 2862	738 3286	233 6811	495 3525
63	53 030	1 800	33 9420	724 5216	255 2594	530 7378
64	51 230	1 889	36 8786	709 5244	276 2988	566 7744
65	49 341	1 980	40 1236	693 2079	296 6076	603 3997

HOMANS' MORTALITY AS ADOPTED BY NEW YORK (continued).

	l_x	d_x	Mortality.	log l_x	log d_x	log Mortality.
66	47 361	2 070	43 7025	675 4209	315 9269	640 5060
67	45 291	2 158	47 6508	656 0119	334 0822	678 0703
68	43 133	2 243	51 9994	634 8097	350 8078	715 9981
69	40 890	2 321	56 7730	611 6171	365 7591	754 1420
70	38 569	2 391	61 9963	586 2384	378 6040	792 3656
71	36 178	2 448	67 6590	558 4446	388 7702	830 3256
72	33 730	2 487	73 7300	528 0163	395 6605	867 6442
73	31 243	2 505	80 1925	494 7527	398 8862	904 1335
74	28 738	2 501	87 0398	458 4565	398 1743	939 7178
75	26 237	2 476	94 3663	418 9142	393 7313	974 8171
76	23 761	2 431	102 3227	375 8647	385 8368	009 9721
77	21 330	2 369	111 0741	328 9909	374 6036	045 6127
78	18 961	2 291	120 8167	277 8612	359 9882	082 1270
79	16 670	2 196	131 7523	221 9356	341 6938	119 7582
80	14 474	2 091	144 4461	160 5886	320 2943	159 7057
81	12 383	1 964	158 6294	092 8259	293 2097	200 3838
82	10 419	1 816	174 3376	017 8260	259 2169	241 3909
83	8 603	1 648	191 5709	3.934 6499	216 9794	282 3295
84	6 955	1 470	211 3718	842 2971	167 3442	325 0471
85	5 485	1 292	235 6268	739 1766	111 4012	372 2246
86	4 193	1 114	265 6748	622 5249	046 8753	424 3504
87	3 079	933	302 9386	488 4097	969 7642	481 3545
88	2 146	744	346 8610	331 6297	871 7851	540 1554
89	1 402	555	395 8828	146 7480	744 3147	597 5667
90	847	385	454 1326	2.927 8834	585 0661	657 1827
91	462	246	533 3332	664 6420	391 6407	726 9987
92	216	137	632 9113	334 4538	135 7967	801 3429
93	79	58	732 6007	1.897 6271	762 4944	864 8673
94	21	18	869 5653	322 2193	261 5215	939 3022
95	3	3	1000 0000	0.477 1213	477 1213	000 0000

increased in the rates given by Mr. Fackler, although in three cases the value of log mortality would have indicated a lower figure.

To summarize the conclusions which I believe may be drawn from the existence of this table dating from practically the time when the American Experience Table was adopted as a basis of valuation by the State of New York, the following facts may be considered as established:

1. The American Experience Table is based on a table of the values of r_x to three decimal places for each age from 10 to 95 inclusive. No light is thrown on the method by which this table was derived from the original data.

2. From this table of the values of r_x the value of the logarithm of the rate of mortality was taken out to seven decimal places.

3. Starting with a radix of 100,000 at age 10, the value of log l_{10} was taken out to seven decimal places. The logarithm of the mortality was added to it to form log d_{10}. The natural number of d_{10} was then taken to the nearest unit and subtracted from l_{10} to form l_{11}. A similar process was then followed to form successively log l_{11}, log d_{11}, d_{11} and l_{12}, the process being repeated for each age to the end of the table.

VALUES OF r_x AND ITS DIFFERENCES.

x	r_x	1000 Δr_x	1000 $\Delta^2 r_x$	1000 $\Delta^3 r_x$	x	r_x	1000 Δr_x	1000 $\Delta^2 r_x$	1000 $\Delta^3 r_x$
10	133.500	− 475	− 14	14	55	53.835	−3543	126	13
11	133.025	− 489	0	− 1	56	50.292	−3417	139	3
12	132.536	− 489	− 1	−12	57	46.875	−3278	142	− 1
13	132.047	− 490	− 13	− 5	58	43.597	−3136	141	17
14	131.557	− 503	− 18	−16	59	40.461	−2995	158	13
15	131.054	− 521	− 34	−13	60	37.466	−2837	171	− 6
16	130.533	− 555	− 47	3	61	34.629	−2666	165	−10
17	129.978	− 602	− 44	8	62	31.963	−2501	155	− 2
18	129.376	− 646	− 36	− 8	63	29.462	−2346	153	− 1
19	128.730	− 682	− 44	− 3	64	27.116	−2193	152	− 7
20	128.048	− 726	− 47	3	65	24.923	−2041	145	− 4
21	127.322	− 773	− 44	2	66	22.882	−1896	141	− 3
22	126.549	− 817	− 42	0	67	20.986	−1755	138	− 5
23	125.732	− 859	− 42	− 3	68	19.231	−1617	133	1
24	124.873	− 901	− 45	− 4	69	17.614	−1484	134	− 1
25	123.972	− 946	− 49	− 3	70	16.130	−1350	133	− 9
26	123.026	− 995	− 52	− 6	71	14.780	−1217	124	−12
27	122.031	−1047	− 58	− 6	72	13.563	−1093	112	−23
28	120.984	−1105	− 64	− 4	73	12.470	− 981	89	−21
29	119.879	−1169	− 68	− 6	74	11.489	− 892	68	−14
30	118.710	−1237	− 74	− 5	75	10.597	− 824	54	−10
31	117.473	−1311	− 79	− 8	76	9.773	− 770	44	− 5
32	116.162	−1390	− 87	−13	77	9.003	− 726	39	−19
33	114.772	−1477	−100	− 9	78	8.277	− 687	20	28
34	113.295	−1577	−109	− 7	79	7.590	− 667	48	3
35	111.718	−1686	−116	0	80	6.923	− 619	51	1
36	110.032	−1802	−116	9	81	6.304	− 568	52	−25
37	108.230	−1918	−107	5	82	5.736	− 516	27	−25
38	106.312	−2025	−102	− 7	83	5.220	− 489	2	5
39	104.287	−2127	−109	−11	84	4.731	− 487	7	10
40	102.160	−2236	−120	−24	85	4.244	− 480	17	28
41	99.924	−2356	−144	−25	86	3.764	− 463	45	16
42	97.568	−2500	−169	− 1	87	3.301	− 418	61	−28
43	95.068	−2669	−170	−24	88	2.883	− 357	33	−36
44	92.399	−2839	−194	14	89	2.526	− 324	− 3	35
45	89.560	−3033	−180	−19	90	2.202	− 327	32	48
46	86.527	−3213	−199	24	91	1.875	− 295	80	−80
47	83.314	−3412	−175	36	92	1.580	− 215	0	65
48	79.902	−3587	−139	66	93	1.365	− 215	65	
49	76.315	−3726	− 73	68	94	1.150	− 150		
50	72.589	−3799	− 5	32	95	1.000			
51	68.790	−3804	27	23					
52	64.986	−3777	50	30					
53	61.209	−3727	80	24					
54	57.482	−3647	104	22					

4. The values for the rate of mortality appearing in this table were constructed from the logarithm of the mortality by taking out the corresponding natural number to seven decimal places.

MR. JOFFE.

(AUTHOR'S REVIEW OF DISCUSSION.)

It is fortunate that the manuscript in Mr. Henderson's possession was saved from the Equitable's fire, and that it fell into the hands

of one who knew how to make good use of it! The conclusions which Mr. Henderson has drawn from the study of the table of "Homans' Mortality as Adopted by New York" seem to be without a flaw.

It goes without saying that I am extremely gratified that Mr. Henderson corroborates the principal point in my hypothesis that the fundamental function in the construction of the American Experience Table is the function r_x, the reciprocal of q_x, *i. e.*, the "number of persons out of which one will die in a year," and furthermore that in his computations Homans took r_x to three decimal places. The reason why I did not compute r_x for ages below 70, as Mr. Henderson has done, lies in the fact that Mr. Fackler's figures of q_x at the lower ages had five significant figures only, and as their reciprocals had the same number of significant figures, it was inadvisable, if not impossible, to draw conclusions as to whether q_x or r_x was the primary function. Mr. Henderson's problem was simplified by the discovery of the q_x column with seven decimal places.

Had Mr. Henderson given the necessary additional details of computation and rearranged the columns in their logical or chronological order in a new table, the first and last lines of that table would have looked somewhat as follows:

(1) x	(2) r_x	(3) $\log r_x$	(4) $\operatorname{colog} r_x = \log q_x'$	(5) $\log l_x = \log$ of (8)	(6) $\log d_x = (4) + (5)$	(7) $d_x = \log^{-1}$ of (6)	(8) l_x $l_{x+1} = l_x - d_x$
10	133.500	2.1254813	$\bar{3}$.8745187	5.0000000	2.8745187	749	100,000
11	133.025	.1239333	.8760667	4.9967349	.8728016	746	99,251
12	132.536	.1223339	.8776661	.9934583	.8711244	743	98,505
...	.....	.....	.....	.....	.....	.....	.....
94	1.150	0.0606978	$\bar{1}$.9393022	0.3222193	0.2615215	18	21
95	1.000	0.0000000	0.0000000	0.4771213	0.4771213	3	3

If we should compare this with the table of details given in my second paper,* we would find that column (4) above corresponds with the previous column (5), column (5) with column (3), column (6) with column (6), column (7) with (7), and that l_x in column (8) is computed in the same manner as l_x was computed by me in column (9). The principal difference in our methods is that Mr. Henderson deals with a radix of 100,000, while I commenced with a radix of 10,000,000, and then reduced it to 100,000 by dividing d_x' by 100. It will be recalled that the adoption of the 10,000,000 radix for the American Experience Table was suggested in Mr. Fackler's "Remarks," as one that would "correspond to its author's original ideas." However, the material recently discovered by Mr. Henderson seems to prove that the American Experience Table can also be constructed with a radix of 100,000.

* *T. A. S. A.*, XIV, 36–37.

As we are not dealing in abstract probabilities and are not endeavoring to find out whether a certain method cannot reproduce the American Experience Table, our only aim being to ascertain the method actually pursued by its author, it is necessary carefully to appraise the relative value of the facts presented by Mr. Fackler in his "Remarks" and of the conclusions drawn by Mr. Henderson from the manuscript table of "Homans' Mortality as Adopted by New York." Mr. Henderson does not mention whether the manuscript table is accompanied by any text or bears any date. However, the expression "Adopted by New York" indicates the lower limit for such a date: since the New York State Law, adopting the American Experience Table as a standard, was passed on May 6, 1868, it is evident that the manuscript table must have been written after that date, that is, a number of years after the American Experience Table was constructed. We have, so far, no proofs produced that it was written by Homans himself, or that it was copied from Homans' working sheets. The interval between the time when this manuscript table was written and the time of Mr. Fackler's "Remarks" before the National Insurance Convention in 1871 is comparatively so short that the priority of the first could not have exceptionally great weight. On the other hand, the fact that Mr. Fackler spoke at the Convention, where Homans was present, and that Homans did not contradict or correct Mr. Fackler, either at the Convention or, to my knowledge, after the Convention, would indicate that there is sufficient ground for assuming that the American Experience Table was originally constructed with a 10,000,000 radix.

We are therefore confronted with a dilemma. On the one side, in my last paper I succeeded in reproducing all the figures of the American Experience Table without a single exception, on the assumption of a radix of 10,000,000; on the other, Mr. Henderson now shows how the American Experience Table can be deduced directly from the smaller radix of 100,000. I am the last to claim that a certain method is the best or the only correct method, simply because that method has been employed by me, and I am ready to admit that, taking all in all, it seems to me that Mr. Henderson's recent discovery tends to show that the original radix was indeed 100,000. However, a certain amount of caution in this case would not be inadvisable. I wish to repeat what I said when giving my details of computation in my second paper:† "it appears to me that until the values of q_x of the American Experience Table are reproduced just as perfectly by another method, which, for some reason, would appear to be more plausible, or until some one proves the present method to be defective, we shall be justified in believing that Homans not only could, but actually did, use this very method in his computations." A similar reservation will not be out of place, when accepting the whole series of figures of r_x to 3 decimal places,

† *T. A. S. A.*, XIV, 30.

as computed by Mr. Henderson. A new discovery, bringing to light some new facts, may compel us in the future to revise the method propounded by Mr. Henderson, as we are now ready to modify the method which I have employed. But this should not discourage us in our investigation. We are following the inevitable course of evolution: here additional facts change a hypothesis into a theory, and there a would-be theory is discarded because it contradicts newly discovered facts.

However, even granting that Mr. Henderson's figures of r_x are entirely and absolutely the same figures as those employed by Homans, the problem before us is still only half solved, as it is necessary to find the method and manner in which the values of r_x were computed by Homans. That these values could have been obtained through a graphic method, by reading them off as ordinates of the r_x curve to 3 decimal places, seems to me beyond the limits of human powers, both as to the faculties of the eye and as to the fineness of the instruments of measurement. To all appearances, this unsolved part of the problem may prove even more difficult than the corresponding problem in the case of Homans' Table No. 1, with which I dealt in my first paper,‡ since in that table, constructed from the 15 years' experience of The Mutual Life, we have the complete columns of the "unadjusted values" as published by Homans, and the column corresponding to r_x is given by him in detail to 2 decimal places only.

We must of course discount such broad statements as were made by Superintendent Barnes in his New York State Insurance Report for 1868, to the effect that Homans "availed himself of all the other statistics at hand (in addition to the actual experience of The Mutual Life) for ascertaining the laws of mortality applicable to healthy insured lives in this country," and that "all the standard European Tables were also used for the purpose of comparison in graduating and modulating the American." We know that these statements were made mainly for "reasons of state," in connection with the celebrated controversy between the Superintendent of New York and the Commissioner of Massachusetts. Nevertheless, we cannot deny that Homans based the construction of the American Experience Table on a modification of his Table No. 1, and before attempting to find Homans' method of computing the r_x of the American Experience Table to 3 decimal places, it may prove helpful to return to the problem (which I set to myself but did not succeed in solving a few years ago) of computing the r_x of his Table No. 1 to 2 decimal places, for which we have the full data of "unadjusted values."

So much for the main issue under consideration. There are two minor points which may be worth mentioning:

1. In comparing the logarithms of q_x, as computed from his values of r_x, with those quoted in the manuscript table, Mr. Henderson

‡ *T. A. S. A.*, XII, 258, first paragraph.

thought he had found three exceptions in the ages: 24, 29 and 35, but he pointed out that the "marginal tables in books of logarithms then in use were not always as complete as those now given." By using Callet's "Tables Portatives de Logarithmes," edition of 1795, I find the following: For age 24, .3 of 348, which equals 104.4, is counted in the table of proportional parts as 105, so that log 124.873 $=2.0964686$, and therefore log $q_{24}=$ colog $r_{24}=\bar{3}.9035314$, just exactly as in the manuscript table. Similarly, for age 29, .9 of 362, which equals 325.8, is counted in the table of proportional parts as 327, and for age 35, .8 of 388, which equals 310.4, is counted as 311. Hence, log $r_{29}=2.0787432$, and log $q_{29}=\bar{3}.9212568$, while log $q_{35}=\bar{3}.9518768$,—in both of these cases again the last figure is exactly the same as in the "log mortality" of the manuscript. Thus we see that Mr. Henderson's figures of r_x conform with "log mortality" without any exception.

2. Should one go to the trouble of comparing the figures of the "mortality" column in the manuscript with those appearing in the column headed "reciprocal" in my paper,* he would find a unit-difference in the last place in the following 6 instances:

Age.	"Reciprocal."			"Mortality."
80	.1444460	instead	of	.1444461.
82	.1743375	instead	of	.1743376.
87	.3029385	instead	of	.3029386.
88	.3468609	instead	of	.3468610.
91	.5333333	instead	of	.5333332.
92	.6329114	instead	of	.6329113.

The difference is due to the fact that the former values are obtained directly as reciprocals of the corresponding r_x, while, in order to arrive at the latter, we pass through the logarithm, co-logarithm and anti-logarithm. If we take, for instance, age 91, $r_{91}=1.875$ $=1\frac{7}{8}$, hence $q_{91}=1/r_{91}=8/15$, $=$the circulating fraction $.5\dot{3}$; but when finding this value by means of logarithms as $\log^{-1}$ colog r_{91}, we arrive at .5333332.

It may be well in conclusion to revert for a moment to a few lines of the address which Homans made at the Society's first dinner. Referring to his predecessor (Gill), Homans says: "He drew with a free pencil a curve showing for each age the number out of which one person would die, and on that hypothetical basis constructed certain rates of premiums, etc." But when speaking of the method used by himself in constructing the American Experience Table, Homans tells us only this: "I drew a curve representing the approximate rates of mortality at different ages: and then found, by a simple method of adjustment, the rates of mortality now called the 'American Experience Table.'" Nothing is said here with reference to the "number out of which one person would die" (that

* *T. A. S. A.*, XIV, 32.

is, the function which I have denoted by r_x), although judging from his previous statement that he "had nothing to do but to follow the line which he (Gill) had so well defined," one would expect Homans to have used Gill's method, with r_x as the fundamental function. Who knows whether we shall ever be able to restore that "simple method of adjustment" which was employed by Homans, and be certain that that indeed is *the* method? So far a careful examination of the minutes of the meetings of the Insurance Committee of the Board of Trustees of The Mutual Life has not yielded anything bearing upon the purely mathematical or actuarial side of the American Experience Table, although many facts of interest have recently been brought out concerning the historical development of the Table, and perhaps, on some future occasion, portions of those minutes may be reproduced in the *Transactions* for the information of the Actuarial profession.

EFFECT ON MORTALITY OF SELECTION BY THE INSURED; MORTALITY AFTER THE DEFERRED DIVIDEND PERIOD—ARTHUR HUNTER.

VOL. XIV, PAGE 38.

WRITTEN DISCUSSION.

MR. HENDERSON:

This paper deals with a subject which is of great importance to many of us, owing to the large number of policies originally issued on the deferred dividend plan that are now reaching the ends of their period. Perhaps the most valuable comment which can be made on this paper is to show to what extent the experience of the Equitable Life Assurance Society confirms the experience of the New York Life. Conditions in these two companies with regard to settlements at the end of the period and percentage of policies continued thereafter were substantially the same, and it is interesting to note that the experience regarding adverse selection was also very similar. The experience of the Equitable was taken out a few years ago by policies, covering substantially the same classes as those mentioned in Mr. Hunter's paper.

The following table gives a condensed summary of the post-tontine experience of the Equitable, in comparison with the experience by policies of the New York Life analyzed in the same way:

It will be noted that the experience in the different classes is relatively much the same in the two companies, the percentage being, as a rule, a little higher in the case of the Equitable than in the case of the New York Life.

This experience covers post-tontine policies in all countries where the northern premium rate is charged. It would appear, as a result of this experience, that the practice of making due provision for the additional mortality due to adverse selection at the end of the deferred dividend period, when calculating the dividend then apportioned, is fully justified.

Policy Year After End of Period.	New York Life.			Equitable Life.		
	Actual.	Expected.	Percentage.	Actual.	Expected.	Percentage.
Ordinary Life Continued for Original Amount.						
1–5	453	340	133%	932	690	135%
6 and over	184	182	101	832	748	110
1-end	637	522	122	1764	1438	123

Policy Year After End of Period.	New York Life.			Equitable Life.		
	Actual.	Expected.	Percentage.	Actual.	Expected.	Percentage.
Limited Payment Life Continued for Original Amount.						
1–5	213	184	116%	271	216	125%
6 and over	75	77	97	91	92	99
1-end	288	261	110	362	308	118
Paid Up with Evidence of Insurability.						
1–5	41	47	87%	95	148	64%
6 and over	26	39	67	99	127	78
1-end	67	86	78	194	275	71
Paid Up without Evidence of Insurability.						
1–5	187	195	96%	181	178	102%
6 and over	88	98	90	197	211	93
1-end	275	293	94	378	389	97

MR. HUTCHESON:

We are much indebted to Mr. Hunter for having given us the results of the investigation which he made into the mortality experienced by those holders of 10-, 15- and 20-year deferred dividend policies in the New York Life Insurance Company who continued their policies in force after the end of the deferred periods.

In 1851, long before deferred dividends were thought of, Mr. J. A. Higham called attention to the fact that "those who dropped their policies were those who permanently enjoyed good health; while those who had fallen into bad health were careful to retain them" (*J. I. A.*, I, 191). He there states that attention had not previously been called to this point so far as he knew. Ever since, there have been some who contended that those who surrendered were bad risks, while others have argued on the other side of the question, but there can be no doubt, from the figures furnished us by Mr. Hunter, that those who surrender their policies at the end of the deferred dividend periods, when cash values are guaranteed, are better risks than those who continue their contracts. This is the first time that figures such as Mr. Hunter's have been published, and the Society is again indebted to him for bringing another new matter to its attention.

Mr. Hunter states that the mortality rate depends partly upon the percentage of policyholders who continue their policies in force, which is another way of stating that the good lives surrender their policies and the bad lives continue theirs. It is, therefore, a matter of great importance to trace the proportion of policies which are continued beyond the end of the deferred dividend periods. He

states that, in the case of his company, about 25 per cent. of the deferred dividend policies reaching the end of their deferred dividend periods between the years 1894 and 1909, were continued in force, whereas about 65 per cent. of such policies reaching the end of their periods after 1909, were so continued. The figures given in his tables on page 40 are the results experienced amongst the 25 per cent. of the policies continued in force, and it is, of course, anticipated that the experience amongst policies maturing after 1909 will show a smaller percentage of actual deaths in excess of the expected.

It would be interesting to compare the corresponding lapse and mortality rates in other companies, but while I am able to give a little information regarding the lapse rate in The Mutual Life, I cannot furnish the mortality rates, as our experience amongst the so-called "deferred dividend" policies does not go back far enough to give us sufficient data for the purpose. The great majority of our deferred dividend policies were issued on the 20-year distribution plan, and very few of these have reached the end of their 20-year distribution periods more than five years ago.

It is interesting, however, to state our experience regarding discontinuance at the end of the deferred dividend period. Mr. Hunter states that the rate of discontinuance in his company was about 75 per cent., whereas the rate in this company, for the smaller amount of experience maturing within the last few years, was about half the rate given by Mr. Hunter. I understand that it was the practice of the New York Life to send a statement to each policyholder just before the end of the deferred period, advising him, inter alia, of the cash surrender value obtainable at the end of the period. This, no doubt, induced the policyholders in many cases to surrender their contracts. The Mutual Life never sent such notices quoting the cash values, and this probably accounts for the smaller rate of discontinuance—under 20 per cent. in the case of 10-year distribution policies, about 30 per cent. in the case of 15-year distribution policies, and under 40 per cent. in the case of 20-year distribution policies, or, taking into account the different amounts insured under the three plans, probably about the same combined rate as that experienced by the New York Life since 1909.

Of course, in selling these deferred dividend policies, the cash values obtainable at the end of the deferred dividend periods were brought prominently before the policyholders, and it is probable that many of these policies will be surrendered at the end of the periods, no matter what the companies may do to discourage the policyholders. The fact stated by Mr. Hunter, that the discontinuance rate has been decreased from 75 per cent. to 35 per cent., shows what can be done when the companies endeavor to have such policies kept in force. It will be instructive to hear what other companies have done towards this end and the results which they have attained.

It may be interesting to some members of the Society to know

what The Mutual Life did some years ago to prepare itself for the extra mortality which it anticipated would be experienced after the end of the deferred periods. We anticipated that a considerable proportion of those holding 15- and 20-year distribution policies who were in good health at the end of the dividend periods, would then surrender their policies, and that a large proportion of those who were in bad health, would continue their contracts. In order to meet the resulting heavy mortality after the deferred dividend periods, the company, acting on the advice of Mr. Emory McClintock, its actuary, decided to gradually build up reserves in excess of the reserves ordinarily held, the American $3\frac{1}{2}$ per cent. reserves. These special reserves are held for 15- and 20-year distribution policies issued during the years 1899 to 1906 inclusive. As these extra reserves are held for all such policies, it was determined to allow the policies to be surrendered at the end of these dividend periods for the full amounts of reserve then held, and guarantees are contained in the policies to this effect.

The assumption made regarding mortality *after the deferred dividend periods* was that it would agree with that shown by Jones' Tropical Table (*T. A. S. A.*, III, 299) for ages over 55, and that, for ages under 55 at maturity, the rate would increase from $133\frac{1}{3}$ per cent. of the American at age 35 gradually up to the Jones' rate at age 55, where it is about 170 per cent. of the American.

The table on page 40 of Mr. Hunter's paper shows the following percentages of actual mortality to expected by the American Table after the end of the deferred periods.

Policy Year after Def. Period.	Ratio of Actual to Expected Claims.
1	184 per cent.
2–3	123 per cent.
4–5	119 per cent.
6–10	106 per cent.
11–17	107 per cent.
Total	126 per cent.

It is evident, therefore, that the assumptions as to mortality after the deferred dividend period made by the Mutual for such policies are conservative.

PROVISION FOR EXPENSES IN SAVINGS CONTRACTS—MILES M. DAWSON.

VOL. XIV, PAGE 44.

WRITTEN DISCUSSION.

MR. MOIR:

I have read this paper three or four times with the idea of getting the author's viewpoint, and the article seems to boil down to a strong plea for greater economy, especially in connection with limited payment and endowment policies. With this plea I have the heartiest sympathy; and, if we consider the paper strictly from an ethical standpoint, we shall find it interesting to discuss the opinion expressed by Mr. Dawson and summarized in his last paragraph, namely, *that no inroads can properly be made for the payment of expenses upon the principal of savings.* In earlier parts of the paper special emphasis is laid upon *initial* expenses on such contracts being unjustifiable and wrong. When expressed thus directly, I am afraid I cannot agree with the author on the ethical question. The most common investment in this country is probably real estate. When a purchaser wishes a specific piece of property and employs a real estate broker to effect the purchase, he always has to pay legal expenses, title insurance, etc., sometimes also a commission to his broker. Initial expense is clearly present. When bonds are put on the market, the bond managers always receive a commission on the sale; when placed through a syndicate, the syndicate gets the bonds at a lower price than that at which they are sold to the public; the difference is composed of two parts, namely, (1) insurance premium for risk incurred in case the bond issue may not be taken by the public, and (2) expenses of sale. Even when we come right down to the simplest savings bank investment, we find that a depositor cannot withdraw money immediately after deposit without losing all the interest on what he has paid in; moreover, the interest received on a savings bank deposit is always less than that which the savings bank earns; and when the immense sums in savings banks which earn no interest at all are taken into consideration, we shall find that Mr. Dawson's statement: "Such a necessary charge, including investment losses, ranges from $\frac{1}{8}$ per cent. to $\frac{1}{4}$ per cent. in well managed companies" is scarcely correct. The fact that some well managed savings banks pay 3 per cent. or $3\frac{1}{2}$ per cent. interest, and others 4 per cent. is in itself an indication of a greater range than is stated. The expenses must be met, and whether these expenses are in the nature of a heavier initial pay-

ment or a continuous payment during the period of investment, is altogether *immaterial to the ethical question, being only a matter of commercial expediency.*

Passing to the question of life insurance, Mr. Dawson says that the straight modified preliminary term valuation is "compatible with entirely justifiable conditions." Personally I go further than this, and say that under present conditions in many parts of America the 20 payment life modification is "justifiable." But this attitude on my part is not based upon ethical considerations, but is a practical dictum dependent solely upon present day conditions. Moreover, I hope that sometime, in the not very distant future, the expense of procuring new business may be so much reduced that even the straight modified preliminary term method will cease to be justifiable. I hope that education in life insurance will be so advanced that a good agent may write a million of direct business in a year where now he can write barely a third of that amount. If that condition should develop, the first year's expense will doubtless be less, and the mode of valuation now justifiable will then become, to use Mr. Dawson's expression, "indefensible"; not because "principles of professional ethics should be applied to the problem," as Mr. Dawson says, but because economic conditions have changed.

The first part of Mr. Dawson's paper beclouds the principal issue to which I have directed attention in the foregoing, because of the accusations which are made. References to "unconscionable fraud," or "fraudulent misrepresentations," or, again, the poor investor being "lied to about gains from lapses, etc.," and of the actuary's skill being utilized "to confuse and bedevil this manifestly correct principle," or of the actuary's "certificate of solvency" being applied "to bolster up a solemn farce,"—all of these expletives distract the reader's attention from the principles which justify the writing of the paper. Again, even if we were to concede Mr. Dawson's general position, I, for one, cannot see why an ordinary life policy should be treated, either ethically or theoretically, as in a different position from a long term endowment or limited payment. To this extent the argument lacks consistency. Again, the comparison between what is the *best* practice in Great Britain with what is known to be the *worst* practice in the United States is unfair, and all the more unfair when the strong expressions appearing in this paper are used.

A personal argument often appeals more strongly than direct reasoning. Mr. Dawson says: "It is manifest that men would not with full knowledge invest their savings unless lied to about gains from lapses, etc., in other words, induced by fraudulent misrepresentations."

I think I know reasonably well what a life insurance policy is, and Mr. Dawson's statements would lead one to suppose that no actuary, or "man of sound mind" knowing true conditions, would take out any form of policy other than an ordinary life in the United States. Personally, I carry endowment policies and limited pay-

ment policies, both in British and American companies, and not a dollar's worth of ordinary life insurance. I took my insurance with full knowledge of the facts. This seems to me the simplest way of showing that such strong language has only a limited application, to say the least of it.

Moréover, the entire paper ignores practical conditions as they exist, especially the laws of the various states, and the tax requirements imposed as the result of impecuniosity rather than with any desire to place a burden upon life insurance. So long as a single or endowment premium is taxed 2 per cent. or more, I fail to see how any sensible company can do otherwise than impose a percentage loading to meet such charges.

As our civilization progresses the number of middlemen is increased. This is a necessity of progress in luxury. A wholesale tea merchant could make a much better bargain in buying vegetables if he went to a farmer who needed tea and effected a direct exchange—yet he sells his tea for twenty-five cents a pound, knowing that the farmer will pay fifty cents; and he buys vegetables from a retail dealer, knowing that he pays nearly double the price the farmer received. The principle is so fundamental that it seems almost foolish to remind a body of scientists that if certain commodities are supplied in an attractive form, with the minimum of trouble to the buyer, then a better price can be obtained—hence the large increase in the sale of package goods, although people generally know that they pay higher prices than they would if buying their supplies in bulk.

In like manner, if endowment insurance proves more attractive for saving purposes than a savings bank, it will be freely sold, and the salesman or agent must receive such compensation as his qualities would earn in other directions. We can only keep good men in the agency field of life insurance by paying them as much as they could earn elsewhere. This phase of the question is purely economic—supply and demand—and it needs no warm language to support or condemn it.

MR. RHODES:

The principle which Mr. Dawson desires to have us accept is stated in his own words to be that no inroads can properly be made for the payment of expenses upon the principal of savings deposited or invested, and he would have it apply to all payments to a life insurance company in excess of its life premium.

Elizur Wright in his "Savings Bank Life Insurance," published in 1872, argued that a company must treat every premium on a policy extending for more than one year as if a certain part of it, more or less, were a mere savings bank or trust-fund deposit, with only this difference from an ordinary one: that it cannot be withdrawn until the death of the insured or the stipulated termination of the policy. Mr. Wright held that the cost of collecting premiums may as fairly be proportioned to their amount in this business as

any other, whether they are devoted to insurance or self-insurance. He also held that all the other working expenses, including the cost of procuring business, should be assessed in proportion to the insurance value of the policies for the year in which they occur, because that value represents the comparative interest which each member has in the company, as an insurance company, in that year. The insurance value, as Mr. Wright expressed it, is the present value of the future normal costs of insurance.

It will be observed that Mr. Wright's view differed from Mr. Dawson's in two respects. He argued that the savings feature was found in continuous level premium life policies, and he thought it right to assess certain expenses against the entire premium on all forms of insurance. I am inclined to question whether Mr. Dawson would apply his idea to annuities and pure endowments, although these appear to be included in the statement in the last paragraph of his paper referred to heretofore, and it would seem that if he yielded this point his argument would fall with it.

While I am in full sympathy with Mr. Dawson insofar as the general result which he desires to accomplish is concerned, I cannot see that his suggestions regarding commissions, surrender values, and loadings are necessary. I do not know why commissions, either first year or renewal, should be the same in amount, for the same amount of insurance, at a given age, upon the life, limited payment, or endowment plan. I have a preference for the European plan of paying the same amount of commissions, regardless of age or plan; but such a plan cannot be adopted in this country by any one company without embarrassment. For practical reasons the commission scale of any one company must be in some degree proportioned to those of its competitors, and the company doing business in New York must meet the requirements of Section 97 of the New York Insurance Law. Recognizing the imperfections of the commission scales in general use in this country, the Mutual Benefit some few years ago abandoned the same and adopted a new scale, which has been very satisfactory in its working. Under the new plan, the maximum percentage of commissions is paid on ordinary life policies issued up to about age 50, after which the percentage is gradually reduced, so that at the oldest age the percentage is less than half what it is at the younger ages. On all other plans, except term policies, the commission is the same in amount as that for an ordinary life policy issued at the same age, plus a small percentage of the difference between the premium on the endowment or limited premium life policy and the premium on the ordinary life policy, the additional percentage being diminished as the difference between the premiums increases. The same amount of commissions is paid, whether premiums be paid anually, semi-annually or quarterly. The plan has had the effect of throwing the weight of the business upon the ordinary life plan at the younger ages. I do not claim that the plan is ideal, but I am of

the opinion that it overcomes, as far as is now practicable by a single company, the objections to the plans in general use in this country, and I do not think it is open to any serious ethical objection.

The statement that our laws provide for the same surrender charge, no matter what method of reserves has been employed, does not appear to me to be strictly correct. It is true that laws generally provide that the surrender charge shall not exceed two and one half per cent. of the amount insured, beginning at the end of the third policy year. A company is free to adopt any basis of surrender values which it chooses, within this limitation. So far as I know, it has not been demonstrated that this surrender charge is not sufficient for any company whose plans of doing business are such that it is reasonably entitled to the patronage of the insuring public.

There appears to be no good reason why companies who desire to do so, may not, under the present plan, accumulate special reserves to cover expenses on paid-up policies. Some American companies do so, I believe, at the present time.

MR. DAWSON:

(AUTHOR'S REVIEW OF DISCUSSION.)

I much regret that there should not be a spontaneous discussion of the topic of my paper, following the two written criticisms which have been presented. Of course, I do not assume, in the absence of such discussion, that all or most of the members of the Society are necessarily in harmony with all that I have said. It would be surprising if a paper such as this should not at first blush strike some of the members as ultra-progressive and as presenting a thing in which they were not ready immediately to acquiesce. It should be remembered, however, that it would not be a valuable paper upon a new line of investigation if such were not the case. It must break new ground.

It is true, as Mr. Moir says, that there are some strong expressions in my paper. He has, however, unintentionally misapprehended the connections in which they appear. The things which I have condemned are, I am sure, such as all the actuaries belonging to the Society would themselves unsparingly condemn.

For instance, the expression "the actuary's skill may be utilized to confuse and bedevil this manifestly correct principle" is employed in the first part of my paper regarding a proposal to invite working-men and others to deposit their savings in institutions supposed to be similar to savings banks, whereas all the money deposited during the first two years is to be taken for expense purposes. Immediately after using this expression, I proceeded to explain how some actuaries in the United States had in times past "confused and bedeviled" this manifestly correct principle, (viz.:

that the principal amount deposited in a savings bank or similar institution where the money is merely to be held and re-invested, must not be trenched upon for expenses), by certifying that because the present value of the entire future payments to be made upon such saving certificates is, together with the present assets, sufficient to guarantee the return of the bare principal paid, the company is solvent. If any of you can think of language which better describes what the actuary is, in such case, invited to do, I am willing to adopt it and substitute it in my papers.

In Mr. Moir's criticism he refers to the following proposition set forth in my paper:

"If so much of the interest actually received by the concern upon its investment, *i. e.,* 4 per cent., for instance, out of the 5 per cent. or even 6 per cent. realized, were so appropriated as realized, it is manifest that men would not with full knowledge 'invest' their savings, unless lied to about 'gains from lapses,' etc. —in other words, induced by fraudulent misrepresentations."

As regards this he takes issue, because he has personally purchased endowment policies instead of whole life policies, the distinction between which is discussed at a much later point in my paper. The language which he quotes was used where I was discussing a pure investment of savings in an institution similar to a savings bank, and I do not think these expressions are more severe than such conduct calls for.

The words, "unconscionable fraud," to which Mr. Moir also refers, are to be found on page 46, immediately following a further statement concerning these purely savings contracts sold under false pretenses by institutions certified to be solvent merely because the present value of their future receipts, plus their present assets, is sufficient to enable them to guarantee the return of principal deposited with them.

But one of the expressions referred to, "abuses of a flagrant character," was used in connection with anything which even purported to be life insurance. These abuses, starting on page 53, are enumerated, and I shall be glad to have the members of the Society consider as to whether or not the expression is not fully justified. In such connection it must be remembered that my paper set out by showing that persons defending utterly indefensible practices in savings institutions similar to saving banks were citing the practices of life insurance companies in justification.

I am glad that attention was called to these expressions, since it certainly would have been unfortunate for the impression to become widespread that condemnation, such as this, is intended to be visited upon those who might differ with me as to the fundamental distinction between life policies, on the one hand, and limited payment and endowment policies, on the other. That is a matter for discussion and through consideration, and not one about which I mean to be dogmatic.

Mr. Moir is in error in supposing that I intended to put forward a plea for economy. I regard that as entirely a separate question. The question here is as to the point where abstraction of funds, intended to be for investment only, commences. It would nowadays be generally allowed, I think, by actuaries that the use, for instance, of the entire first premium on a 10-year endowment policy for current mortality and expenses is not justified. The question is where the line should be drawn.

It is not at all impossible that there might be extravagance, even when the line is drawn properly as regards this principle; since, for instance, if it were established that the ordinary life is to be considered the basis for determining this matter, ordinary life insurance might be obtained and conducted at an excessive rate of expense, and this would not in the least affect that matter.

Mr. Moir criticises the following statement in my paper:

"It being rightly regarded by genuine savings institutions as vicious in the extreme either to impair the principal paid in, or even to diminish the interest, in order to remunerate agents for inducing men to 'save' their money in such a way."

He expresses the opinion that this is not necessarily true, and cites investments in real estate and in the bonds of corporations, in both of which cases there may be, and usually are, expenses. But this is a different case from the deposit of money in a savings bank or similar institution, which is made upon the basis that the principal deposited is to be safely kept and is to be returned, either without interest or with such interest as may be allowed. Of course, if money is borrowed, the borrower has a right to do with it what he pleases, unless restrained by some condition of the contract. The lender depends upon the promise to repay with interest, and upon the security offered, if any; but in the case of a deposit in a savings bank he depends only upon the promise to return his money with or without interest according as he may withdraw at one time or another, and the solvency of such a bank never is, and never ought to be, certified unless it has on hand assets equal to the entire amount of such deposit with all interest which has actually been credited.

As regards savings deposits purely, I am confident that no member of the Society will dispute the correctness of this position.

Both my critics do call in question, though they offer nothing to take its place, my further proposition, which is that the apparent excuse which life insurance practices offer for conduct such as they were cited to defend by the French actuary, is found in failure to recognize that a fundamental distinction should be made by actuaries between premiums on whole life policies and the additional amounts collected by them in the cases of limited premium life policies and endowment policies.

Mr. Rhodes refers to the proposal made by the late Elizur Wright, which was that the loading for expenses be a function of the cost of insurance so that each member would contribute toward the pay-

ment of the company's expenses in exact proportion to the cost of the insurance under his policy, which would itself be computed, of course, on a natural premium basis.

Now, if the normal thing in life insurance which people actually would purchase for the purpose of protecting their families, were it sold for what it is, were natural premium insurance, I think the position of Mr. Wright would be justified. The truth is, as is well known, however, that natural premium insurance cannot be sold; although in the form of increasing premium insurance after a limited number of years merging into level rate whole life insurance, a limited sale can be found for it. In other words, it is not the thing which men understand by life insurance.

In the same way, renewable term insurance cannot be sold upon a purely renewable term basis. It also must merge into a whole life level premium policy at some ages, and indeed, as experience has shown, in order to be at all readily saleable, must carry the suggestion that there will be little or no increase in the actual rates, the increase being largely offset by surplus carried over from one term to another. It follows, therefore, that this is not the thing which men ordinarily understand by life insurance and will readily purchase as such.

The thing which they, undoubtedly, do understand, is the level premium whole life insurance; and this may, I think, fairly be taken as the normal, natural thing for a man to purchase who has in mind nothing but the protection of those dependent upon him now and for the future.

My proposal re-stated is, that, as regards this whole life insurance, investment is really incidental and for the purpose merely of maintaining the insurance, and is not necessarily a thing present in the mind of the policyholder at all. He is concerned that the company keep itself solvent and be ready to maintain his rate level, and that, consequently, it should accumulate whatever funds are required to assure this. The company is justified in computing its premiums for this policy, taking into account both the amount and the incidence of all of its expenditures, whether for expenses or in the payment of claims; and in adjusting the equities of the policyholder, as, also, if it choose, in computing its reserves, it is likewise justified in taking into account both the amount and the incidence of all of its expenditures.

But when we come to the limited payment or endowment policy, unquestionably the policyholder is distinctly invited to invest additional money over and above the ordinary price for life insurance purchased purely for protection. In one case, he is asked to invest this money in order to relieve himself from the payment of premiums after a term of years; or, in other words, to prepay those premiums. In the other case, he is asked to invest it in order to mature the policy during his lifetime; or, in other words, as a pure investment.

Now, under the circumstances and taking into account that the practices of life insurance companies are cited as justification for utterly indefensible practices on the part of savings institutions similar to savings banks, is there not here a question which fairly calls for full and careful consideration on the part of the actuary as to whether the principle should not be established that nothing should be taken from the principal of such additional payments to be applied to the payment of expenses? This should be studied. If it is not the right criterion, nobody will be happier to have the right criterion found than the man who presented this paper.

Mr. Moir has expressed the hope, while dwelling on the question of economy, that insurance may some time be procured at a cost materially lower than the New York and New Jersey standards now provide for. This is a separate question; and as I have pointed out, the principle which I have presented for discussion has nothing to do with the question of economy in the conduct of the life insurance business. But from my examination of the business in other countries, as well as our own, I must say that there appears to be no reasonable basis for that hope or expectation. There are individual companies which have succeeded in procuring their business more cheaply, but in general they do not do so in any country.

Another point is raised by Mr. Moir, viz.: that the companies are required to pay taxes on their entire premium income, and that it would be unfair not to charge the taxes against the premiums which were taxed. With this I concur. It is obvious that no other course would be justifiable.

The personal statement which Mr. Moir makes, that notwithstanding the fact that he knew some of the additional money over and above the ordinary life rate had been used for expenses, he himself purchased limited payment and endowment insurance, is, I think, beside the mark. I might say the same of myself, also; but we both knew that in the actual management of the company, including its distribution of surplus, such policies were, on the whole, treated fairly, and my argument concerning abuses and the ethical questions arising out of them is merely that what is fair treatment should be more accurately discerned and outlined in theory and realized in practice.

Mr. Moir's suggestion that the complexity of our civilization and of its needs justifies large expenditures, cannot be considered to be applicable to the case of a depositor in a savings bank; and I think has no application to the fundamental proposition that a savings institution similar to a savings bank is not actually solvent unless it has on hand at all times assets equal to the principal invested and all credited interest.

Mr. Rhodes asks whether my idea would be applicable to annuities and endowments. As regards the former, it is not, strictly speaking, an investment, since, for instance, the entire amount paid

may be lost; the annuitant has no interest in the matter except that the company shall accumulate such funds as are necessary to enable it to pay the annuities. Pure endowments are not often sold, but if they were, the same principles would appear to apply as in the case of an annuity.

I have not given as much attention to the problem in connection with this, as in connection with life insurance, and forbear to express further opinion at this time.

It is much to be hoped that this entire subject will continue to attract attention, and that many papers may be written upon the same, to the end that in all its bearings what is right and best may appear. I have no personal interest that the particular view which I have brought forward shall prevail; but have brought it forward simply and purely because it appealed to me as the correct distinction, which will probably be found to be fully justifiable.

ORAL DISCUSSION.

MR. MOIR: I am very glad indeed that the criticisms by Mr. Rhodes and myself brought forth such a full reply from Mr. Dawson, because it puts a different complexion on the paper from what it had, not only to me, but also to some other members of the Society. I tried to read the paper as a whole, and naturally took the remarks as being applied to actuaries in general. I am sure Mr. Dawson agrees with me that the members of this Society stand strongly against misrepresentation, and condemn most of the things to which he calls attention. Personally, I am at one with Mr. Dawson as regards most of these objectionable practices, although I do not agree with the ethical proposition which he put forth.

MR. PAPPS: I am very much surprised to hear Mr. Dawson say that these references are not supposed to apply to life insurance, but to other forms of investments. And the summary of the paper, as I remember it—I haven't a copy before me—refers in a number of paragraphs—I think in every single one—to life insurance, and I think Mr. Dawson unwittingly, in that way, may have misled some of his readers.

MR. DAWSON: What I attempted to present in my paper was, first of all, a perfectly clear proposition that, in connection with purely investment contracts, not involving life contingencies at all, where the proceeds of the investment were to be re-invested and the only way in which the savings were to be accumulated was by that re-investment, any deduction from the principal for expense purposes, thereby impairing the principal, is a thing to be condemned by actuaries, and a thing which actuaries would not, if they understood it, support by giving certificates of solvency to such an institution. This assertion by me was brought out by the attitude

taken by a French actuary, that such a practice in connection with savings is justified by practices in life insurance. Against that practice in connection with savings and the attempted justification of it by reference to practices in life insurance, I launched the condemnation which has here been referred to; and the reading of my paper will show that it was those things which are there specifically condemned.

Now, it was from that proposition, in which I am quite of the opinion that practically the entire actuarial profession in this country and abroad will concur and do concur, that I proceeded to make an examination and a very brief investigation as to how such a reference to practices in life insurance had even an appearance of justification; and in that I endeavored to work out certain principles by which it would be possible to distinguish even slight departures in life insurance from correct practices. As to that I claim nothing, except that I did try honestly to make such an investigation, and did honestly write down the things which seemed to me to determine slight departures from correct practices that are not, and generally have not, been recognized.

In the latter part of the paper, which has also been referred to by one of my critics, I named a number of departures, I think no one of which has been recognized as a correct line of conduct by the actuarial profession generally, but all of which have, notwithstanding, been made in our own country at different times, and some which I referred to as having happened recently. These were not reflections upon the actuarial profession, because the actuarial profession in the United States has not stood for such things; but they were reflections upon practices which have at times marred life insurance in our own country, and which grew out of failure fully and clearly to recognize correct principles.

Now, as regards the exact application which I there referred to, of recognizing a whole life policy as the norm, and characterizing deposits in addition to a whole life premium as essentially savings, I claim nothing except that this should be carefully studied and examined with a view to determining whether it is correct, and if it be not correct, what is correct and should be our guide in those matters. Mr. McClintock, in this country, and Mr. Sprague, in Great Britain, took this position as regards minimum reserves long ago. We should consider whether it is wisely taken and, if so, whether it should not be extended to other matters.

But one word more concerning endowment insurance,—and this is to illustrate how far we have travelled along the line towards what seems to me an absolutely correct principle, and its application in our country. As most of you know, I was a field man before I took up actuarial work. While in the field I was once confronted, when endeavoring to get the application of a young married man in business for $5,000 protection for his family, by an agent who proposed he should take a $1,000 ten-year endowment on

the estimate of nearly four and a half per cent. compound interest at the end of ten years. Upon inquiring, I found the company for which that agent was working was paying a commission of 75 per cent. of the ten-year endowment premium, as large a commission as it would have paid on the whole life premium. That was one of the largest companies in this country, and its actuaries were expecting reserves to be put up as on the level premium basis. This practice had grown up independent of the actuary, and was one which he could not then by any means prevent. Now, we have grown away from that. To-day there is not a man within the sound of my voice who would not condemn it as loudly as I condemn a like practice in connection with pure savings contracts not involving life insurance at all. And all this paper is intended to do—and if it will do it, I shall forever rejoice that I placed it before you, notwithstanding the misunderstanding it may have caused concerning my own position,—is to give us something to think about, and perhaps thereby call forth other papers which will throw further light upon this difficult and exceedingly important subject.

TOTAL DISABILITY BENEFITS; WITH SPECIAL REFERENCE TO PENSION FUND METHODS OF CALCULATION—JAMES F. LITTLE.

VOL. XIV, PAGE 55.

WRITTEN DISCUSSION.

MR. JACKSON:

The method of dealing with the subject employed by Mr. Little, namely that of treating it from the pension fund aspect, was suggested by Mr. Hunter in his recent paper, but was not made use of by him.

We are, therefore, greatly indebted to Mr. Little for his careful elaboration of the method and the great number of formulas used by him, covering nearly every form of benefit and plan in which it may arise. This painstaking exposition should be of the greatest use to students.

I am, however, compelled to take issue with Mr. Little on one point. On page 62, he assumes that if the insured have the option of selecting paid-up insurance or an annuity settlement, that when disablement is the result of disease, and death is practically certain within a few years, the instalment option will not be selected. While this is the course which would be pursued theoretically, I do not think it would be found to be the case in actual practice, any more than the assumption that impaired lives do not surrender their policies.

I am presenting herewith the experience in disability claims of a Southern company operating only in the South, with which I was recently connected. The company began business in 1905, and in 1912 amalgamated with two others and terminated its separate existence.

In issues from 1905 to 1909, the permanently disabled had the option of a paid-up policy, or of receiving the face of the policy in ten annual instalments, the balance to be payable to the beneficiary or commuted if death took place before the ten instalments were paid. In 1909 and the early part of 1910, the benefit consisted of a paid-up policy only. During the latter part of 1910 and afterwards, the benefit consisted of the two options, as in the early years of the company. The accompanying schedule contains the experience of the company.

Of the seventeen claims, fourteen had the privilege of selecting either a paid-up policy or ten instalments. Eleven selected the latter, and three the former. Of the eleven who selected the instalment option, seven died within one year; one lived three years;

SCHEDULE OF TOTAL AND PERMANENT DISABILITY CLAIMS APPROVED.

Claim.	Date of Issue.	Age at Issue.	Amount of Insurance.	Date of Approval of Claim.	Date of Death or Recovery.	Mode of Settlement.	Sex.	Cause of Disability.
1	12/1/06	53	$3,000	11/29/07	4/18/08	Instal.	Male	Double Pneumonia.
2	10/15/05	28	2,000	6/29/08	6/15/09	Instal.	Female	Tuberculosis.
3	2/1/08	39	1,000	10/27/08	11/28/11	Instal.	Male	Insanity.
4	1/1/08	26	1,000	6/16/09	12/11/09	Instal.	Male	Tuberculosis.
5	1/15/08	30	1,000	7/9/09	2/17/10	Instal.	Male	Tuberculosis of Hip Joint and Lungs.
6	3/1/06	38	1,000	8/5/09		Instal.	Female	Articular Rheumatism.
7	10/1/06	46	2,000	9/20/09	9/5/11	Pd.-Up In.	Male	Dementia.
8	11/1/05	28	4,000	2/21/10	10/7/10	Instal.	Male	Tuberculosis.
9	3/15/08	46	2,000	3/12/10		Instal.	Female	Locomotor Ataxia.
10	8/15/07	45	5,000	4/19/10		Instal.	Male	Heart Disease.
11	12/1/06	25	1,000	1/10/11	2/7/11	Instal.	Male	Tuberculosis.
12	1/15/08	55	1,000	3/14/11	9/17/11 Recovered	Instal.	Male	Spinal Sclerosis.
13	5/15/09	26	4,000	5/15/11	10/15/11	Pd.-Up In.	Male	Paralysis.
14	4/1/07	39	1,000	6/–/11	7/28/11 Recovered	Pd.-Up In.	Female	Pellagra.
15	4/1/10	25	1,000	8/1/11	5/20/12	Pd.-Up In.	Male	Consumption.
16	8/15/06	56	1,000	7/15/11	8/7/11	Pd.-Up In.	Male	Uraemia.
17	3/1/09	32	1,000	10/23/11	8/–/13	Pd.-Up In.	Male	Tuberculosis.

and three still survive. It will be noticed that one died within one month. The brother of this man was an insurance agent, and corresponded with the company concerning the selection, and was given a table showing the commuted value at a high rate of interest during ten years. In spite of this information, and the knowledge that death would take place shortly, he accepted the instalment option.

Claim No. 8 was also an insurance agent and knew well what commutation of the instalments meant, yet selected the immediate benefit of $400 a year for ten years in preference to a paid-up policy for $4,000.

It will be noticed that 2 and 3, who were entitled to the paid-up only, recovered. I have no doubt that they would have taken the instalments, had they possessed the right. I may mention that No. 13, a doctor, who recovered, surrendered his policies the next year.

This experience, I think, shows that many of the permanently disabled prefer an immediate benefit, which is often very acceptable in times of sickness, to the greater benefit payable to their beneficiaries after their deaths.

ORAL DISCUSSION

MR. DAWSON: Mr. President, I was asked to prepare a written discussion of Mr. Little's paper, but I was unable to do so, owing to the burden of other duties. I would not feel satisfied to prepare a written discussion, unless I took up all the technical parts of the paper. But I do feel that I ought not to neglect saying something concerning the paper.

I think Mr. Little has made a very valuable contribution to our literature relative to this subject. I am reminded that, when we first took it up, there was very little literature in the English language of any importance relating to total and permanent disability, or calculations concerning it. My friend, Herr Schaertling of Switzerland, was the author of a series of valuable formulas upon the subject; and the practice of furnishing benefits of this character had become very extensive in Germany and German-speaking countries, and had already spread to some extent in continental Europe; but, when first introduced into the United States, it appeared to have taken the form of giving these benefits without making any calculations for reserve or extra premium.

Since that time, through the efforts of several of the members of the Society, nearly all of them among the junior members,—and by junior members I do not mean necessarily the younger members, but those who have more recently become members,—under the encouragement of older members, including myself, we have made great progress, until today I think we have almost, if not quite, the most valuable literature relating to total disability benefits in the world, in the *Transactions* of our own Society.

Mr. Little's contribution is of especial value, not only as regards total and permanent disability, but also as regards disability benefits in general, for the following reasons:

The British way of dealing with the subject in connection with friendly societies, was to group in an aggregate form without regard to years of duration. That grew out of the actual nature and practice of the friendly societies of Great Britain, the opinion really being that, when a member joined at a certain age, he should be charged a premium or assessment, sufficient to take care of sicknesses happening among members of his age and at ages above his own as he successively attains them, including those sicknesses which were already in progress at his age at the time he joined. In consequence of that, in handling those friendly societies, the usual custom, in fact almost the universal custom, has been to merge the reserve for taking care of persons who are under disability, with the reserve for maintaining the level rates for life.

The first computations of this character, which were made in this country, were, so far as I know, made in my office, and were precisely of that type.

The paper now presented by Mr. Little is in furtherance of a suggestion, sufficiently worked out for practical purposes, perhaps, by Mr. Hunter in one of his papers, namely, that the insurance departments of the United States, on the whole, would not understand the mingling of claim reserves and policy reserves involved in the British friendly society practice. Consequently, Mr. Little has brought forward in this paper another practice which you will also find in Great Britain in connection with computations made by actuaries of pension funds maintained by employers, or by employers and employees jointly, which are applicable to pensions earned on account of premature disability as well as by superannuation; that idea being that claim reserves, or the reserves to provide for the pensions on account of premature disability, should be set up. And, in consequence, he has deduced a number of formulas, some of which possibly might be simplified to advantage, for use in making calculations of that character.

I have already stated that I have not had time to prepare a technical criticism. I think the best form for a technical criticism in any event probably will be future papers prepared by members of the Society. But in breaking the ground, or at least further breaking the ground, which Mr. Hunter's paper initiated, I think Mr. Little's work will prove to be of great service to the members of the Society, and particularly to those who wish to carry forward investigations of this problem.

MR. LITTLE.

(AUTHOR'S REVIEW OF DISCUSSIONS.)

Mr. Dawson has suggested that some of the formulas might be simplified. I am rather pessimistic on this point, as I found the

complexity of the formulas bearing strict relation to that of the benefits.

Mr. Coburn has pointed out an error which I regret to say has crept into the formulas of Section XVII. The first paragraph of text in this section should be amended to read as follows:

"XVII. In the case of endowment insurances the benefit differs from that under whole life policies in that all payments under the special annuity falling due from $x+t$ must be excluded. The value of payments for life in respect of disability occurring between ages x and $x+t-k$ is, as per Section XV*A*:

$$\frac{d}{k\overline{D}_x^{aa}}\,{}^{rs}_{k}M_x - ({}^{rs}_{k}M_{x+t-k}).$$

For the interval between ages $x+t-k$ and $x+t$, as the annuity does not increase after age $x+t-1$, the method of Section IX applies, giving a value (treating the payments as continuing for life) of

$$\frac{d}{k\overline{D}_x^{aa}}\{{}^{rIa}M_{x+t-k} - {}^{rIa}M_{x+t} - S^i_{x+t}({}^{rs}M_{x+t-k} - {}^{rs}M_{x+t})\},$$

It remains only to find an expression for the value of the payments falling due from age $x+t$, in respect of disability occurring prior thereto. The value sought is, per unit of first payment of annuity, . . ."

Formula (34) consequently becomes

$$\overline{{}^{t}|{}_{k}}A_{x:\overline{t}|} = \{{}^{rs}_{k}M_x - {}^{rs}_{k}M_{x+t-k} + {}^{rIa}M_{x+t-k} - {}^{rIa}M_{x+t}$$

$$-S^i_{x+t}({}^{rs}M_{x+t-k} - {}^{rs}M_{x+t}) - N^i_{x+t}(k^{rs}M_x - {}^{rs}R_{x+t-k} + {}^{rs}R_{x+t})\}\,\frac{d}{k}\,\frac{1}{\overline{D}_x^{aa}}.$$

Formula (35) is not affected, but formula 35*A* becomes

$$\overline{{}^{t}|{}_{k}}A_{x:\overline{y-x}|} = \{{}^{rs}M_x - {}^{rs}M_{x+t-k} + {}^{rIa}M_{x+t-k} - {}^{rIa}M_y - S^i_{x+t}({}^{rs}M_{x+t-k}$$

$$-{}^{rs}M_y) - N^i_{x+t}(k^{rs}M_x - \overline{x+t-y}\,{}^{rs}M_y - {}^{rs}R_{x+t-k} + {}^{rs}R_y)\}\,\frac{d}{k}\,\frac{1}{\overline{D}_x^{aa}}.$$

The formula in Section XXII concerning the corresponding reserves is correct.

I regret that there was no discussion of my system of notation, as I had hoped for some useful suggestions thereon.

Book Notices.

Medico-Actuarial Mortality Investigation. Volume III. Effect of Occupation on Mortality. Compiled and Published by the Association of Life Insurance Medical Directors and the Actuarial Society of America, New York, 1913. Pp. 219.

Reference having already been made in the review of Volumes I and II, it is not necessary to repeat the general scope of the Investigation. This particular volume (III) deals with the mortality in various occupations. The occupations selected were divided into two groups—"Required" and "Optional." Those "Required" were selected as ones about which there was considerable demand for information, while the "Optional" group constituted those which were not considered of much importance by the majority of the companies, or for which the committee did not believe that a sufficient amount of data existed.

The mortality ratios for each class apply to the occupation at the time of application, as it was impracticable to take account of subsequent changes.

All cases in each occupation were considered, whether policies were issued at regular rates or specially rated in any way, except such cases where there was any other impairment than that incident to the particular occupation. As a matter of interest, however, the effect of minor impairments was investigated as follows:

(*a*) Normal weights in nineteen "Required" occupations.

(*b*) Normal weights in thirteen "Optional" occupations.

(*c*) Overweights in six "Required" occupations.

This investigation was only general, and as it confirmed the impression that applicants subject to minor impairments were less freely accepted when engaged in unfavorable or doubtful occupations, it was decided not to carry it further.

In finding the expected deaths in each occupation the M. A. Table was used, a copy of which is given on page 27.

The results under each occupation are discussed, more or less in detail, on pages 8 to 25, and comparisons are drawn between the death rate per hundred from specific causes supposed to be attributable to the particular occupation, and general death rates or other experiences.

A convenient General Summary showing "Actual Deaths," "Expected Deaths" and "Rates of Actual to Expected Deaths," by occupations, is given on pages 28–31.

The rest of the volume is taken up with the details of the various classes, showing "Insurance Years," "Age Groups," and "Synopses."

The same general caution is applicable to the results contained in this volume as to other similar investigations; that is, that where the numbers are small the ratios must not be looked upon as conclusive, and also that in the case of doubtful occupations the risks assumed have been more than ordinarily carefully selected.

Insurance and the State. W. F. Gephart, Ph.D. New York, MacMillan Co., 1913. Pp. xiii, 228.

This treatise takes up in a systematic manner the relations of insurance to the state. The author first investigates the character of insurance, following this with a historical review of state insurance in various countries. He then takes up the question whether the state should monopolize life insurance, giving consideration to the objects which it may be proposed to accomplish by such monopoly. These purposes he classifies as

1. As a Source of Revenue.
2. For Social Developmental Purposes.
3. As a Complete Method of Regulation.

The author next takes up the subject of fire insurance and analyses it in the same way. The last three chapters of the book are devoted to a similar discussion of social or workmen's compensation insurance and sickness, old age and unemployment insurance. The analysis of the subjects treated in this book appears to be very careful and fairly complete. One notices, however, a marked omission in that no reference whatever is made to the experiment in government life insurance which has been carried on for many years in New Zealand.

Report on the Graduation of Ages. George King. London, 1913. Pp. 28.

This report, which was made to the Registrar-General by the author, covers the graduation of the following census returns:

1. The enumerated population of England and Wales subdivided into males and females, and second, as regards each sex, into unmarried, married and widowed.

2. The unmarried population of the County of London similarly subdivided, but as regards the County of London it was not thought necessary to deal separately with the subdivision of the sex into unmarried, married and widowed.

3. The enumerated population of certain selected districts, subdivided into males and females. The method adopted by the author in making this graduation is, so far as the body of the work is concerned, the method which was explained by him in Volume XLIII of the *Journal of the Institute of Actuaries*. Special methods were necessary, however, in connection with the infantile ages and some parts of the analyzed data which are fully explained in the report. An appendix gives the graduated numbers and a comparison of the graduated results with the ungraduated.

Abstract from the Minutes of the Semi-annual Meeting of the Actuarial Society of America, held in Boston, on Thursday and Friday, October 23 and 24, 1913.

Copley-Plaza Hotel, Boston,
October 23, 1913.

FIRST DAY.

The meeting was called to order by the Vice-President, Mr. Arthur Hunter, at 11 o'clock A. M. The following members were present:

FELLOWS.

Blehl,
Brinkerhoff,
Carpenter,
Cooper,
Craig, J. D.,
Craig, J. M.,
Cushman,
Dawson,
Dow,
Fackler, E. B.,
Ferguson,
Fitzgerald,
Flynn,
Hardcastle,
Henderson,
Hunter, A.,
Hutcheson,
Ireland,
Jackson,
Johnson,
Johnston,
Laird,
Little,
McKechnie,
Macaulay,
Marshall, E. P.,
Moir,
Morris,
Mowbray,
Papps,
Portch,
Rhodes,
Rose,
Sheppard,
Strong, W. M.,
Torrey,
Vineberg,
Watt,
Welch,
Wells,
Wood, A. B.,
Wood, W. A. P.,
Woodward, G. B.,
Woodward, J. H.,
Wright.

ASSOCIATES.

Cameron,
Dickenson,
Flanigan,
Franks,
Gibb,
Gould,
Graham, G.,
Hall, A. F.,
Halliday,
Howe,
Kaufman,
King,
Linton,
Marshall, E. W.,
Milligan,
Parker,
Phillips,
Rice,
Sinclair,
Smith, V. R.,
Speers,
Strong, A. W.,
Tebbetts,
Thomas,
Walker,
Washburn, J. H.,
Washburne, A. C.,
Whitney.

The minutes of the annual meeting held in New York May 15 and 16 were approved as printed in the *Transactions*.

The Secretary reported that the Council had appointed Mr. W. M. Strong to act as Treasurer during the time of the meeting on account of the absence of Mr. Alsop.

The Society then proceeded to the reading of papers prepared for this meeting.

The Society adjourned at 1 o'clock and reassembled at 2:30 with Vice-President J. M. Craig in the chair.

After the reading of the papers was completed, the Society took up the discussion of the papers presented at the meeting in May, 1913.

At 5 o'clock the Society adjourned to meet at dinner at 7 o'clock and to reassemble for business on the following morning at 10 o'clock.

SECOND DAY.

October 24, 1913.

The meeting was called to order by Vice-president J. M. Craig acting as Chairman. The discussion of papers was proceeded with, and after this discussion had been completed an informal discussion on the results of the Medico-Actuarial Experience was held with Vice-president A. Hunter in the chair.

On motion duly seconded, the following resolution was unanimously adopted:

Resolved, that the thanks of the Society be given to our hosts, the Actuaries of Boston, for their thoughtful preparations and very successful arrangements for the Fall Meeting of the Actuarial Society and the entertainment of the ladies accompanying the members; with assurances that their generous hospitality has been fully appreciated and much enjoyed.

EXAMINATIONS.

The following is a list of those who passed the examinations held by the Society on May 21 and 22, 1913.

FELLOWSHIP.

Part II.

Cook, H. M.,
Johnson, M. L.,
MacFarlane, W.,
MacLean, J. B.,
Vineberg, H. E.

Part I.

Cameron, J. C.,
Coburn, A.,
Kaufman, H. N.,
Linton, M. A.,
Marshall, E. W.,
Pequegnat, A. E.,
Sinclair, C. C.,
Stephenson, H. R.

ASSOCIATESHIP.

Section B—Parts I and II.

Cameron, W. J.,
Holmes, H.,
Hurd, H. G.,
Leslie, W.,
Milligan, S.,
Tebbetts, J. W.,
Watson, A. D.

Section B—Part II.

Flanigan, J. E.,
Rydgren, A. A.,
Little, R. (Part I not yet passed).

Section B—Part I.

West, C. J.

Section B—Part I and Section A—Part II.

Buchanan, J. D.

Section A—Parts I and II.

Larus, J. R.,
Michelbacher, G. F.,
Pettegrew, A. A.,
Rutherford, C. D.

Section A—Part II.

Gamerdinger, C. W.,
Green, W. W.,
Hanson, H. O.,
Hill, E. B.,
Koeppe, A. J.,
Mattocks, R. L.,
Porter, L. A.,
Shepperd, C. O.,
Williamson, W. R.

SECTION A—PART II. (Part I not yet passed.)

Bosworth, L. A., Madrill, J. D.,
Wheeler, R. A.

SECTION A—PART I. (Part II already passed.)

Blackadar, W. L., Beecroft, J. D.,
MacCharles, F. D.

SECTION A—PART I.

Bagley, W. N., Hartwell, G. W.,
Bissell, C. R., Kioke, K.,
Barber, W. P., Reichgott, H.,
Elston, J. E., Tarbell, T. F.

HORTON WOODS COCHNOWER.

Horton Woods Cochnower, an Associate Member of this Society, died on the 14th day of September, 1913, in the thirty-sixth year of his age. Mr. Cochnower began his insurance career in the office of the Union Central Life Insurance Company. He was of a quiet, retiring disposition, but was a very close student, and had he lived would doubtless have made his mark in his chosen profession. At the time of his death Mr. Cochnower was the Actuary and Secretary of the Guarantee Life Insurance Company of Houston, Texas, and was highly esteemed by his fellow officers of that company.

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